

### Inverses Exercises 1

- (1) (a) Assume  $A^{-1} \exists$ , solve  $BA = C$  for  $B$  (square matrices )  
(b) Assume  $A^{-1}, B^{-1} \exists$  and  $ACB = ADB$ , prove  $C = D$  (square matrices )  
(c) Solve  $B = P^{-1} A P$  for  $A$  (square matrices )
- (2) Assume  $A, B, C, D$  are square matrices and  $a$  is a scalar.  
State assumptions that allow one to solve for  $B$ , then solve for  $B$ .
- (a)  $AB + aB = D$  ; (b)  $AB + 2C = DB$  ; (c)  $AB = C + 3B$   
(d)  $BA + BC = C$  ; (e)  $BA = CA$
- (3) Assume  $A, B, C, D$  are square and invertible. Find the inverse of  
(a)  $ABCD$  ; (b)  $ABA^{-1}B^{-1}$
- (4) Assume  $A^{-1} \exists$ , find the inverse of (a)  $A^3$ ; (b)  $3A$ ; (c)  $-A^4$ ; (d)  $(A^{-1})^2$
- (5) Find  $A^{-1}$  if (a)  $A^2 - 2A - 3I = 0$  ; (b)  $A^6 - 5A^4 + 3A^2 + 2I = 0$
- (6) (a) Assume  $B \neq 0$  and  $AB = 0$ , can  $A^{-1} \exists$  ?  
(b) Assume  $A^2 = 0$  and  $A \neq 0$ , can  $A^{-1} \exists$  ?  
What if  $A = 0$ ? Does  $A^{-1}$  exist in that case ?  
(c) Assume  $P^2 = P$  and  $P \neq I$ , can  $P^{-1} \exists$  ?

---

### Answers :

- (1a)  $CA^{-1}$  ; (1c)  $PBP^{-1}$  ; (2a)  $(A + aI)^{-1} D$  where  $(A + aI)^{-1}$  must  $\exists$  ;  
(2b)  $(-2)(A - D)^{-1} C$  or  $2(D - A)^{-1} C$  where  $(A - D)^{-1}$  must  $\exists$  ;  
(2c)  $(A - 3I)^{-1} C$  where  $(A - 3I)^{-1}$  must  $\exists$  ; (2d)  $C(A + C)^{-1}$  where  $(A + C)^{-1}$  must  $\exists$  ;  
(2e)  $C$  where  $A^{-1} \exists$   
(3a)  $D^{-1} C^{-1} B^{-1} A^{-1}$  ; (3b)  $BAB^{-1}A^{-1}$   
(4a)  $(A^3)^{-1} = (A^{-1})^3 = A^{-3}$  ; (4b)  $\frac{1}{3}A^{-1}$  ; (4c)  $-(A^{-1})^4 = -(A^4)^{-1} = -A^{-4}$  ; (4d)  $A^2$   
(5a)  $\frac{1}{3}(A - 2I)$  ; (5b)  $\frac{1}{2}(5A^3 - A^5 - 3A)$

## Inverses Exercise 2

(1) Let  $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}$ ,  $C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(a) Verify that  $(AB)^{-1} = B^{-1}A^{-1}$ ; (b) Verify that  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(2) Find  $A$  given that

(a)  $A^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$ ; (b)  $(7A)^{-1} = \begin{pmatrix} -3 & 7 \\ 1 & -2 \end{pmatrix}$ ; (c)  $(5A^t)^{-1} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$ ;

(d)  $(I + 2A)^{-1} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$

(3) Let  $A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$ , show that  $(A^{-1})^3 = (A^3)^{-1}$

(4) (a) Given  $A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ , compute  $A^{-1}$ . (b) Verify that  $AA^{-1} = I$

(c) Write the system  $\begin{cases} 2x + 3y + 4z = -1 \\ 4x + 3y + z = 5 \\ x + 2y + 4z = -2 \end{cases}$  in the form  $AX = B$  or  $A\vec{x} = \vec{b}$  and use  $A^{-1}$  to solve the system.

(c) Write the system  $\begin{cases} 2x + 3y + 4z = 1 \\ 4x + 3y + z = 2 \\ x + 2y + 4z = 0 \end{cases}$  in the form  $AX = B$  or  $A\vec{x} = \vec{b}$  and use  $A^{-1}$  to solve the system.

(c) Write the system  $\begin{cases} 2x + 3y + 4z = 0 \\ 4x + 3y + z = 0 \\ x + 2y + 4z = 0 \end{cases}$  in the form  $AX = B$  or  $A\vec{x} = \vec{b}$  and use  $A^{-1}$  to solve the system.

(f) If  $B = \begin{pmatrix} 8 & 12 & 16 \\ 16 & 12 & 4 \\ 4 & 8 & 16 \end{pmatrix}$ , write  $B = kA$  ( $k =$  a scalar) and

$B^{-1} = (kA)^{-1}$ , that is find  $B^{-1}$  using  $A^{-1}$

(g) If  $C = \begin{pmatrix} -10/3 & 4/3 & 3 \\ 5 & -4/3 & -14/3 \\ -5/3 & 1/3 & 2 \end{pmatrix}$ , write  $C = kA^{-1}$  ( $k =$  a scalar),

$C = kA^{-1}$  and  $C^{-1} = (kA^{-1})^{-1}$ , that is find  $C^{-1}$  using  $A^{-1}$

## Inverses Exercise 2

(5) Given  $A^{-1} = \begin{pmatrix} 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & -4/5 \\ -2/5 & 1/10 & 1/10 \end{pmatrix}$ , find  $A$ .

(6) Given  $A = \begin{pmatrix} a & a \\ 1-a & 1-a \end{pmatrix}$ , verify that  $A^2 = A$

(7) Let  $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ;  $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ , (a) If  $BC = CB$ , show that  $B$  is of form  $\begin{bmatrix} t & s \\ s & t \end{bmatrix}$ ;

(b) If  $CB = 0$ , show that  $B$  is of form  $\begin{bmatrix} s & t \\ -s & -t \end{bmatrix}$

(8) Find 4 matrices satisfying  $A^2 = I_3$  but  $A \neq I_3$

(9) Assume  $A, B, X$  are square matrices of the same size,  $A^{-1} \exists, B^{-1} \exists$ ; solve for  $X$ .

(a)  $AX = AB$ ; (b)  $AX = BA$ ; (c)  $AX = A + B$ ; (d)  $ABX = A + B$ ;

(e)  $AXB = A + B$ ; (f)  $XAB = A + B$

### Answers :

(2 a)  $A = \frac{1}{13} \begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix}$ ; (2 b)  $A = \frac{1}{7} \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$ ; (2 c)  $A = \frac{1}{5} \begin{pmatrix} -2 & 5 \\ -1 & 3 \end{pmatrix}$ ; (2 d)  $\frac{1}{13} \begin{pmatrix} -9 & 1 \\ 2 & -6 \end{pmatrix}$

(4 a)  $A^{-1} = \frac{1}{5} \begin{pmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{pmatrix}$ ; (4 c)  $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12/5 \\ -7/5 \\ -2/5 \end{pmatrix}$ ;

(4 d)  $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2/5 \\ 7/5 \\ -3/5 \end{pmatrix}$ ; (4 e)  $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$  (trivial solution only)

(4 f)  $B = 4A \Rightarrow B^{-1} = (4A)^{-1} = \frac{1}{4} A^{-1} = \frac{1}{20} \begin{pmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{pmatrix}$ ; (4 g)  $C = \frac{5}{3} A^{-1} \Rightarrow C^{-1} = \left(\frac{5}{3} A^{-1}\right)^{-1} = \frac{3}{5} A$

(5)  $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$ ; (8)  $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ ,  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ , etc.

(9 a)  $X = B$ ; (b)  $X = A^{-1} B A$ ; (c)  $X = I + A^{-1} B$ ; (d)  $X = B^{-1} + B^{-1} A^{-1} B$  or  $B^{-1} + (AB)^{-1} B$

(9 e)  $X = B^{-1} + A^{-1}$ ; (9 f)  $X = AB^{-1} A^{-1} + A^{-1}$  or  $A (AB)^{-1} + A^{-1}$