INSTRUCTIONS: THIS TEST CONTAINS 10 QUESTIONS. EACH QUESTION IS WORTH 10 EIGENPOINTS. MAKE SURE THAT YOU SHOW YOUR WORK TO OBTAIN ANY PARTIAL CREDIT. USE THE REVERSE SIDE OF EACH PAGE IF YOU RUN OUT OF SPACE.

1. Let $A = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

   Find the eigenvalues of A and give a basis for each eigenspace.

2. Given the difference equation
$$a_{n+3} = a_{n+2} - a_{n+1} + a_n$$
   with $a_0 = 2$, $a_1 = 1$, $a_2 = 0$.

   (a) Find a_3, a_4, and a_5.

   (b) Write this difference equation as a linear dynamical system $\mathbf{x}_{n+1} = A\mathbf{x}_n$. What is the initial state vector, $\mathbf{x}_0$, for this system.

   (c) Find $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ (the next three state vectors).

   (d) Find the eigenvalues of A. Give any complex eigenvalues in polar form.

   (e) Draw a time plot of this system (reasonably accurate and clear enough to indicate the general behavior of the system).

3. Let $A = \begin{bmatrix} 1 & 2 \\ k & 4 \end{bmatrix}$.

   (a) For what value(s) of k is A symmetric?

   (b) For what value(s) of k is A not invertible?

   (c) For what value(s) of k is A not diagonalizable?

   (d) For what value(s) of k is $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ an eigenvector of A?

   (e) For what value(s) of k does A have complex eigenvalues.

   (f) For those values of k where A has complex eigenvalues plot the magnitude of these complex eigenvalues as a function of k. (So the horizontal axis is the k axis, the vertical axis is the magnitude of the complex eigenvalues. The domain will be the interval from part

   (g) For what value(s) of k does A have positive eigenvalues (that is, when is A positive definite)? Indicate these values on the graph from the previous part.

4. Given the data

x	-2	-1	0	1	2
y	-2	-1	0	0	3

   (a) Find the least squares line $y = \beta_0 + \beta_1 x$ for this data.

   (b) Find the least squares quadratic $y = \beta_0 + \beta_1 x + \beta_2 x^2$ for this data.

   (c) Find the total least squares line for this data. In this case it is enough to give the slope accurate to 2 decimal places.

5. (a) Prove that if Q is an orthogonal matrix then $\|Q\mathbf{y}\| = \|\mathbf{y}\|$.

   (b) Prove that is λ is an eigenvalue of A then $\lambda + k$ is an eigenvalue of $A + kI$.

   (c) Prove that if A and B are symmetric matrices then AB and BA have the same eigenvalues.

6. In the space of integrable functions defined on the interval $[-1, 1]$ define the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

The first 3 orthogonal Legendre polynomials in this inner product space are $1, x, x^2 - \frac{1}{3}$

Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ k & 0 \le x \le 1/k \\ 0 & \text{if } 1/k < x \end{cases}$ where $k > 1$.

   (a) Plot $f(x)$ when $k = 2$. Plot $f(x)$ when $k = 3$.
   (b) What is $\|f\|$ in this inner product space?
   (c) Find the best approximation to $f(x)$ by a polynomial of degree 1 in this inner product space. What happens to this polynomial as $k \to \infty$?
   (d) Find the best approximation to $f(x)$ by a quadratic polynomial in this inner product space. What happens to this polynomial as $k \to \infty$?

7. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Find the QR factorization of A.

8. In the space of $n \times n$ matrices define the inner product

$$\langle A, B \rangle = \text{trace}(A^T B)$$

   (a) A matrix A is symmetric if $A^T = A$. A matrix B is skew-symmetric if $B^T = -B$. Show that if A is symmetric and B is skew-symmetric then A and B are orthogonal relative to the above inner product.
   (b) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ show that $\mathbf{u}\mathbf{v}^T - \mathbf{v}\mathbf{u}^T$ is skew-symmetric.
   (c) Suppose $\mathbf{u}$ and $\mathbf{v}$ are linearly independent unit vectors in $\mathbb{R}^n$ and that θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Find $\|\mathbf{u}\mathbf{v}^T - \mathbf{v}\mathbf{u}^T\|$ relative to the above inner product. Your answer should be a simple expression in terms of θ.

9. (a) Plot $2x_1^2 + 12x_1x_2 - 7x_2^2 = 10$. Include the principal axes (the axes of symmetry) in your plot.
   (b) Give the coordinates (that is, the x_1, x_2 coordinates) of the points on the graph closest to the origin.

10. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 4 \end{bmatrix}$. Let $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

   (a) Find the best approximation to $\mathbf{y}$ by a vector in V.
   (b) Find the distance from $\mathbf{y}$ to V.
   (c) Find a basis for $V^\perp$.

ANSWERS:

1.

$$\lambda_1 = 2, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 1, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2. $a_3 = 1, a_4 = 2, a_5 = 1$

   $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

   $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

   $\lambda = 1, e^{\pm i\pi/2}$

   The time plot should show steady state oscillation.

3. (a) $k = 2$
   (b) $k = 2$
   (c) $k = -9/8$
   (d) $k = 11/9$
   (e) $k < -9/8$
   (f)
   (g) $-9/8 < k < 2$

4. (a) $y = 11/10x$
   (b) $y = -6/14 + 11/10x + 3/14x^2$
   (c) $y = 1.20x$

5. (a) $\|Q\mathbf{y}\|^2 = (Q\mathbf{y})^T Q\mathbf{y} = \mathbf{y}^T Q^T Q\mathbf{y} = \mathbf{y}^T I\mathbf{y} = \|\mathbf{y}\|^2$
   so $\|Q\mathbf{y}\| = \|\mathbf{y}\|$
   (b) Given $A\mathbf{v} = \lambda\mathbf{v}$ then

$$(A + kI)\mathbf{v} = A\mathbf{v} + kI\mathbf{v} = \lambda\mathbf{v} + kmathbfv = (\lambda + k)\mathbf{v}$$

   (c) Given that $A^T = A$ and $B^T = B$ then

$$|AB - \lambda I| = |(AB - \lambda I)^T| = |B^T A^T - \lambda I| = |BA - \lambda I|$$

   Since AB and BA have the same characteristic polynomial, they have the same eigenvalues.

6. (a)
   (b) k
   (c) $\frac{1}{2} + \frac{3}{4k}x$. This approaches $\frac{1}{2}$ as k becomes infinite.
   (d)

7.

$$\begin{bmatrix} 1/\sqrt{2} & 1\sqrt{6} & 1/sqrt3 \\ -1/sqrt2 & 1/sqrt6 & 1/sqrt3 \\ 0 & -2/sqrt6 & 1/sqrt3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & -1/sqrt2 & 0 \\ 0 & 3/\sqrt{6} & -2/sqrt6 \\ 0 & 0 & 1/sqrt3 \end{bmatrix}$$

8. (a) $\langle A, B\rangle$ is equal to the trace of $A^T B = AB$ and is also equal to the trace of $B^T A = -BA$. But the trace of AB is equal to the trace of BA so the above shows that $\langle A, B\rangle = -\langle A, B\rangle$ and therefore $\langle A, B\rangle = 0$
   (b) $(\mathbf{uv}^T - \mathbf{vu}^T)^T = \mathbf{vu}^T - \mathbf{uv}^T = -(\mathbf{uv}^T - \mathbf{vu}^T)$
   (c) $\sqrt{2}\sin\theta$

9. Graph is a hyperbola with closest points $\pm(4/sqrt10, 2/sqrt10)$

10. (a) $\begin{bmatrix} 1 \\ 4 \\ 4 \\ 3 \end{bmatrix}$

(b) $\sqrt{3}$

(c) $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$