Sums of powers of positive integers[*]

§ 1. **An observation.** — Let

$$\sigma_k = 1^k + 2^k + 3^k + \cdots + n^k,$$

where k, n are non-negative integers; $\sigma_k(n)$ if its dependence on n requires emphasis. That σ_{k-1} is a polynomial of degree k in n is seen by summing $(\nu+1)^k - \nu^k$ over $\nu = 1, 2, 3, \ldots, n$. Let σ_k' denote the derivative of σ_k with respect to n, and observe that

$$\sigma_k(\nu) - \sigma_k(\nu-1) = \nu^k, \qquad \text{and hence} \qquad \sigma_k'(\nu) - \sigma_k'(\nu-1) = k\nu^{k-1}.$$

Summing this last result over $\nu = 1, 2, 3, \ldots, n$ gives

$$\sigma_k'(n) - \sigma_k'(0) = k\sigma_{k-1}(n), \qquad \text{or} \qquad \sigma_k' = k\sigma_{k-1} + B^k, \qquad \text{where} \qquad B^k = \sigma_k'(0).$$

This immediately yields formulæ for the sums σ_k. If $2 \leqslant j \leqslant k+1$, the coefficient of n^j in σ_k is k/j times the coefficient of n^{j-1} in σ_{k-1}, the linear coefficient B^k is determined by $\sigma_k(1) = 1$ and the constant term is zero.

§ 2. **Sample computations.** — The formulæ will be illustrated by computing the polynomials up to σ_6. In the first place, it is plain that $\sigma_0 = 1 + 1 + 1 + \cdots + 1 = n$. Next, by the foregoing observation

$$\sigma_1 = \tfrac{1}{2} \cdot 1n^2 + B^1 n, \qquad \text{and} \qquad 1 = \sigma_1(1) = \tfrac{1}{2} + B^1 \qquad \text{implies that} \qquad B^1 = \tfrac{1}{2}.$$

Therefore $\sigma_1 = \frac{1}{2}n^2 + \frac{1}{2}n$. For the sum of squares,

$$\begin{aligned} \sigma_2 &= \tfrac{2}{3} \cdot \tfrac{1}{2}n^3 + \tfrac{2}{2} \cdot \tfrac{1}{2}n^2 + B^2 n \\ &= \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n, \end{aligned}$$

since $1 = \sigma_2(1) = \frac{1}{3} + \frac{1}{2} + B^2$ implies that $B^2 = \frac{1}{6}$. For the sum of cubes,

$$\begin{aligned} \sigma_3 &= \tfrac{3}{4} \cdot \tfrac{1}{3}n^4 + \tfrac{3}{3} \cdot \tfrac{1}{2}n^3 + \tfrac{3}{2} \cdot \tfrac{1}{6}n^2 + B^3 n \\ &= \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2, \end{aligned}$$

since $1 = \sigma_3(1) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + B^3$ implies that $B^3 = 0$. For the sum of fourth powers,

$$\begin{aligned} \sigma_4 &= \tfrac{4}{5} \cdot \tfrac{1}{4}n^5 + \tfrac{4}{4} \cdot \tfrac{1}{2}n^4 + \tfrac{4}{3} \cdot \tfrac{1}{4}n^3 + \tfrac{4}{2} \cdot 0n^2 + B^4 n \\ &= \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n, \end{aligned}$$

since $1 = \sigma_4(1) = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} + B^4$ implies that $B^4 = -\frac{1}{30}$. For the sum of fifth powers,

$$\begin{aligned} \sigma_5 &= \tfrac{5}{6} \cdot \tfrac{1}{5}n^6 + \tfrac{5}{5} \cdot \tfrac{1}{2}n^5 + \tfrac{5}{4} \cdot \tfrac{1}{3}n^4 + \tfrac{5}{3} \cdot 0n^3 - \tfrac{5}{2} \cdot \tfrac{1}{30}n^2 + B^5 n \\ &= \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2, \end{aligned}$$

since $1 = \sigma_5(1) = \frac{1}{6} + \frac{1}{2} + \frac{5}{12} - \frac{1}{12}$ implies that $B^5 = 0$. For the sum of sixth powers,

$$\begin{aligned} \sigma_6 &= \tfrac{6}{7} \cdot \tfrac{1}{6}n^7 + \tfrac{6}{6} \cdot \tfrac{1}{2}n^6 + \tfrac{6}{5} \cdot \tfrac{5}{12}n^5 + \tfrac{6}{4} \cdot 0n^4 - \tfrac{6}{3} \cdot \tfrac{1}{12}n^3 + \tfrac{6}{2} \cdot 0n^2 + B^6 n \\ &= \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n, \end{aligned}$$

since $1 = \sigma_6(1) = \frac{1}{7} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6} + B^6$ implies that $B^6 = \frac{1}{42}$.

[*] After remarks made by John H. Conway during a public lecture (27 September 2017, Montréal)

§ 3. **Exercises.** — The exercises below involve elementary deductions based on § 1. At certain points it will help to have reflected on the computations in § 2, and to have some familiarity with the expansion of a power of a binomial.

Exercise 1. — Complete the proof that σ_{k-1} is a polynomial of degree k in n.

Exercise 2. — Prove that in σ_{k-1} the coefficient of n^k is $\frac{1}{k}$, and that of n^{k-1} is $\frac{1}{2}$.

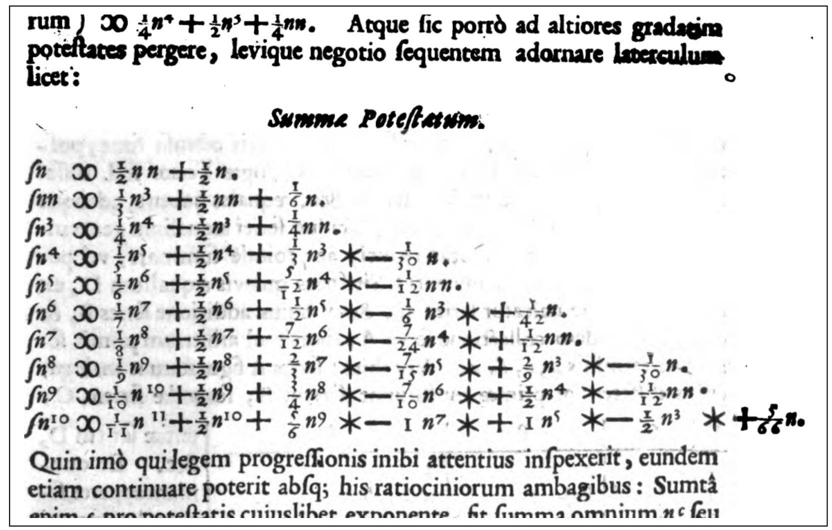
Exercise 3. — Prove that if $k \geqslant 1$, then σ_k is divisible by $n+1$.

Exercise 4. — Prove that $B^\lambda = 0$ if $\lambda > 1$ and λ is odd.
Hint: For $k \geqslant 2$ expand $\sigma_1^k(\nu) - \sigma_1^k(\nu-1)$, express their sum over $\nu = 1, 2, \ldots, n$ in terms of σ_λ, where $k \leqslant \lambda \leqslant 2k-1$ and λ is odd, and explain why these σ_λ are divisible by n^2.

Exercise 5. — Write an explicit formula for the polynomial σ_{k-1} (using $B^\varkappa$, $0 \leqslant \varkappa < k$). (The result suggests a pun, which motivates using a superscript for the index of $B^\varkappa$.)

Exercise 6. — For $k > 1$, express B^k in terms of $B^\varkappa$, for $0 \leqslant \varkappa < k$.

Exercise 7. — The figure below contains an excerpt from a work of Jacob Bernoulli (who wrote $\int n^k$ for σ_k and nn for n^2). Which coefficient in the excerpt is incorrect?


rum) ∞ $\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn$. Atque ſic porrò ad altiores gradatim poteſtates pergere, levique negotio ſequentem adornare laterculum licet:

Summæ Poteſtatum.

$\int n$ ∞ $\frac{1}{2}nn + \frac{1}{2}n.$
$\int nn$ ∞ $\frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n.$
$\int n^3$ ∞ $\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn.$
$\int n^4$ ∞ $\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 * - \frac{1}{30}n.$
$\int n^5$ ∞ $\frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 * - \frac{1}{12}nn.$
$\int n^6$ ∞ $\frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 * - \frac{1}{6}n^3 * + \frac{1}{42}n.$
$\int n^7$ ∞ $\frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 * - \frac{7}{24}n^4 * + \frac{1}{12}nn.$
$\int n^8$ ∞ $\frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 * - \frac{7}{15}n^5 * + \frac{2}{9}n^3 * - \frac{1}{30}n.$
$\int n^9$ ∞ $\frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 * - \frac{7}{10}n^6 * + \frac{1}{2}n^4 * - \frac{1}{12}nn.$
$\int n^{10}$ ∞ $\frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 * - 1n^7 * + 1n^5 * - \frac{1}{2}n^3 * + \frac{5}{66}n.$

Quin imò qui legem progreſſionis inibi attentius inſpexerit, eundem etiam continuare poterit abſq; his ratiociniorum ambagibus: Sumtâ enim c pro poteſtatis cujuslibet exponente, fit ſumma omnium n^c ſeu

[An excerpt from *Ars Conjectandi* (1713), by Jacob Bernoulli]