A comparison principle for improper integrals

Calculus II

201-NYB-05
Comparison principle (vague idea)

A little experience with improper integrals suggests a pair of related, and somewhat vague, expectations.

- An improper integral which is “larger” than a divergent improper integral should be divergent.

- An improper integral which is “smaller” than a convergent improper integral should be convergent.

The comparison principle stated on the next page confirms, in a precise manner, two simple and reasonable elements of these expectations.
Theorem (Comparison principle)

Suppose that the functions \( f \) and \( g \) are continuous on \([\alpha, \infty)\).

a. If \( 0 \leq f(x) \leq g(x) \) for \( \alpha \leq x \), and

\[
\int_{\alpha}^{\infty} f(x) \, dx \text{ is divergent, then } \int_{\alpha}^{\infty} g(x) \, dx \text{ is divergent.}
\]

b. If \( |f(x)| \leq g(x) \) for \( \alpha \leq x \), and

\[
\int_{\alpha}^{\infty} g(x) \, dx \text{ is convergent, then } \int_{\alpha}^{\infty} f(x) \, dx \text{ is convergent.}
\]

The comparison principle is illustrated and explained here.

This note describes and exemplifies the use of the comparison principle.
Strategy

Suppose that $F' = f$ is continuous on $(a, b)$, and that

$$\int_{a}^{b} f(x) \, dx$$

is an improper integral.

It may be possible to use the comparison principle to prove that (*) is convergent, or divergent, without the need to compute $F$. This is convenient if $F$ is complicated (e.g., is not an elementary function).

If (*) is divergent, the result is (typically) a nearly complete analysis.

If (*) is convergent, its value need not emerge from an application of the comparison principle. But the convergence of (*) may countenance manipulations which are otherwise questionable and do yield its value.
A basic strategy involves estimating part of an integrand so that
- the integral can be effectively compared to a simpler integral, and
- the simpler integral can be computed using standard methods, or measured along a standard scale.

Success involves making judgements about the convergence of simplified integrals, to determine which available comparisons are likely to be useful.

If $a < c < b$ and the right side is convergent, then

$$\int_{a}^{b} Af(x) \, dx = A \int_{a}^{c} f(x) \, dx + A \int_{c}^{b} f(x) \, dx.$$

So changing a proper limit of integration, or multiplying an integrand by a non-zero number, does not affect the convergence of an improper integral.
Example 1. Investigate the convergence of \[ \int_{0}^{\frac{1}{3}\pi} \frac{\sec x}{x} \, dx. \]

Since \( 1 < \sec x \) if \( 0 < x < \frac{1}{2}\pi \), it follows that \( 0 < \frac{1}{x} < \frac{\sec x}{x} \) if \( 0 < x \leq \frac{1}{3}\pi \).

Now \[ \int_{0}^{1} \frac{dx}{x} = \lim_{t \to 0^+} \log x \bigg|_{t}^{1} = \lim_{t \to 0^+} \left( -\log t \right) = \infty, \]

so \[ \int_{0}^{\frac{1}{3}\pi} \frac{dx}{x} = \int_{0}^{1} \frac{dx}{x} + \int_{1}^{\frac{1}{3}\pi} \frac{dx}{x} \]

is divergent. \(^{(\ast)}\)

Therefore, the improper integral \[ \int_{0}^{\frac{1}{3}\pi} \frac{\sec x}{x} \, dx \]

is divergent, by the comparison principle.

\(^{(\ast)}\)Henceforth, a change of proper limit or non-zero multiple will not be noted.
Scales

The preceding analysis can be streamlined by referring to the Scale of powers at 0.

The improper integral \( \int_0^1 \frac{dx}{x^p} \) converges if \( p < 1 \), and diverges if \( p \geq 1 \).

This is because

\[
\int_0^1 \frac{dx}{x} = \lim_{t \to 0^+} (-\log t) = \infty,
\]

and if \( p \neq 1 \), then

\[
\int_0^1 \frac{dx}{x^p} = \lim_{t \to 0^+} \frac{1 - t^{1-p}}{1 - p} = \begin{cases} 
\frac{1}{1-p} & \text{if } p < 1, \text{ and} \\
\infty & \text{if } p > 1.
\end{cases}
\]
Example 2. Investigate the convergence of \[ \int_{0}^{\frac{1}{3}\pi} \frac{\sec x}{\sqrt{x}} \, dx. \]

As \( 1 < \sec x \leq 2 \) if \( 0 < x \leq \frac{1}{3}\pi \), it follows that \( 0 < \frac{\sec x}{\sqrt{x}} \leq \frac{2}{\sqrt{x}} \) if \( 0 < x \leq \frac{1}{3}\pi \).

Since the improper integral
\[ \int_{0}^{1} \frac{dx}{\sqrt{x}} \]

is convergent (\( p = \frac{1}{2} < 1 \) in the scale of powers at 0), the improper integral
\[ \int_{0}^{\frac{1}{3}\pi} \frac{\sec x}{\sqrt{x}} \, dx \]

is also convergent, by the comparison principle.
Example 3. Investigate the convergence of \( \int_0^\infty \frac{\pi + e \sin x}{1 + x^2} \, dx \).

Since \(-1 \leq \sin x \leq 1\) for any real number \(x\), and \(e < \pi\), it follows that

\[
0 < \frac{\pi + e \sin x}{1 + x^2} \leq \frac{\pi + e}{1 + x^2},
\]

for all real values of \(x\).

Now

\[
\int_0^\infty \frac{\pi + e}{1 + x^2} \, dx = \lim_{t \to \infty} \left\{ (\pi + e) \arctan t \right\} = \frac{1}{2} (\pi + e) \pi,
\]

so the improper integral \( \int_0^\infty \frac{\pi + e}{1 + x^2} \, dx \) is convergent.

Therefore, the improper integral \( \int_0^\infty \frac{\pi + e \sin x}{1 + x^2} \, dx \) is convergent, by the comparison principle.
Scales (continued)

The preceding analysis can be streamlined by referring to the Scale of powers at $\infty$.

The improper integral $\int_{1}^{\infty} \frac{dx}{x^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

This is because

$$\int_{1}^{\infty} \frac{dx}{x^p} = \lim_{t \to \infty} \log t = \infty,$$

and if $p \neq 1$, then

$$\int_{1}^{\infty} \frac{dx}{x^p} = \lim_{t \to \infty} \frac{t^{1-p} - 1}{1 - p} = \begin{cases} \frac{1}{p - 1} & \text{if } p > 1, \text{ and} \\ \infty & \text{if } p < 1. \end{cases}$$
Example 4. Investigate the convergence of \( \int_{e}^{\infty} \frac{\pi + e \cos x}{(\log x)^3} \, dx \).

Since \( \cos x \geq -1 \) for all real values of \( x \), and (by arithmetic of logarithms) \( 0 < \frac{1}{3} \log x = \log(\sqrt[3]{x}) < \sqrt[3]{x} \), or \( (\log x)^3 < 27x \), for \( x > 1 \), it follows that

\[
0 < \frac{\pi - e}{27x} < \frac{\pi + e \cos x}{(\log x)^3}, \quad \text{if} \quad x > 1.
\]

Since \( \int_{1}^{\infty} \frac{dx}{x} \) is divergent (\( p = 1 \) in the scale of powers at \( \infty \)), the comparison principle implies that the improper integral

\[
\int_{e}^{\infty} \frac{\pi + e \cos x}{(\log x)^3} \, dx
\]

is divergent.
The next three examples involve a basic trigonometric inequality.

In Calculus I the fundamental limit

\[ \lim_{\vartheta \to 0} \frac{\sin \vartheta}{\vartheta} = 1, \]

is deduced from the inequality

\[ \cos \vartheta < \frac{\sin \vartheta}{\vartheta} < 1, \quad \text{if} \quad -\frac{1}{2}\pi < \vartheta < \frac{1}{2}\pi \quad \text{and} \quad \vartheta \neq 0, \]

or equivalently,

\[ \vartheta \cos \vartheta < \sin \vartheta < \vartheta, \quad \text{if} \quad 0 < \vartheta < \frac{1}{2}\pi. \]

(The inequality, for \(0 < \vartheta < \frac{1}{2}\pi\), is seen by comparing the areas of a triangle of base 1 and height \(\sin \vartheta\), a sector of radius 1 and angle \(\vartheta\), and a triangle of base 1 and height \(\tan \vartheta\). The first inequality remains valid if \(-\frac{1}{2}\pi < \vartheta < 0\), since its terms are even functions of \(\vartheta\).)
Example 5. Investigate the convergence of \( \int_{0}^{1} \frac{\sin(x) \cos(1/x)}{\sqrt[9]{x^5}} \, dx \).

If \( 0 < x < \frac{1}{2} \pi \) then \( |\cos(1/x)| \leq 1 \) and \( 0 < (\sin x)/x < 1 \), and hence

\[
\left| \frac{\sin(x) \cos(1/x)}{\sqrt[9]{x^5}} \right| = \frac{\sin x}{x} \cdot \frac{|\cos(1/x)|}{x^{4/5}} < \frac{1}{x^{4/5}}.
\]

Since \( \int_{0}^{1} \frac{dx}{x^{4/5}} \) is convergent (\( p = \frac{4}{5} < 1 \) in the scale of powers at 0),

the comparison principle implies that the improper integral

\[
\int_{0}^{1} \frac{\sin(x) \cos(1/x)}{\sqrt[9]{x^5}} \, dx
\]

is convergent.
Example 6. Investigate the convergence of \[ \int_{0}^{\frac{1}{2}\pi} \log(\sin x) \, dx. \]

If \(0 < x \leq \frac{1}{3}\pi\) then \(\frac{1}{2} x < x \cos x < \sin x < 1\), and so \(\log(\frac{1}{2} x) < \log(\sin x) < 0\).

Now \(\int_{0}^{1} \log(\frac{1}{2} x) \, dx = \log(\frac{1}{2}) - 1 - \lim_{t \to 0^+} \{ t \log(\frac{1}{2} t) - t \} = - \log(2e)\), via partial integration and basic properties of the logarithm.

Hence, the comparison principle implies that the improper integral

\[ \int_{0}^{\frac{1}{2}\pi} \log(\sin x) \, dx \]

is convergent.
Example 7. Investigate the convergence of $\int_{1}^{5} \frac{\arccsc x}{3\sqrt{(x-1)^5}} \, dx$.

If $1 < x < 2$ and $\vartheta = \arccsc x$, then $0 < \vartheta < \frac{1}{3} \pi$, and so

$$\frac{\arccsc x}{3\sqrt{(x-1)^5}} = \frac{2 \cdot \left(\frac{1}{2} \vartheta\right)}{(x-1)^{5/3}} > \frac{2 \sin \left(\frac{1}{2} \vartheta\right)}{(x-1)^{5/3}} = \frac{\sqrt{2} \sqrt{1 - \cos \vartheta}}{(x-1)^{5/3}}$$

$$= \frac{\sqrt{2} \cos \vartheta \sqrt{\sec \vartheta - 1}}{(x-1)^{5/3}} > \frac{\sqrt{x-1}}{(x-1)^{5/3}} = \frac{1}{(x-1)^{7/6}} > 0.$$ 

Since $\int_{1}^{2} \frac{dx}{(x-1)^{7/6}} = \int_{0}^{1} \frac{dx}{x^{7/6}}$ is a divergent improper integral ($p = \frac{7}{6} \geq 1$ in the scale of powers at 0), the comparison principle implies that the improper integral $\int_{1}^{5} \frac{\arccsc x}{3\sqrt{(x-1)^5}} \, dx$ is divergent.
Example 8. Investigate the convergence of \( \int_0^\infty \frac{\sin x}{x} \, dx \).

First of all,

\[
\int_0^\infty \frac{\sin x}{x} \, dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} \, dx + \int_{\frac{\pi}{2}}^{\infty} \frac{\sin x}{x} \, dx,
\]

where the first term on the right side is (essentially) a definite integral because \((\sin x)/x \to 1\) as \(x \to 0\). Partial integration then gives

\[
\int_{\frac{\pi}{2}}^{\infty} \frac{\sin x}{x} \, dx = \lim_{t \to \infty} \left[ -\cos x \right]_{\frac{\pi}{2}}^{t} - \int_{\frac{\pi}{2}}^{\infty} \frac{\cos x}{x^2} \, dx
\]

\[
= \lim_{t \to \infty} -\frac{\cos t}{t} - \int_{\frac{\pi}{2}}^{\infty} \frac{\cos x}{x^2} \, dx.
\]
Example 8, continued.

Since \( \left| \frac{\cos t}{t} \right| \leq \frac{1}{t} \) if \( t > 0 \), \( \lim_{t \to \infty} \frac{-\cos t}{t} = 0 \), so the integral \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \) converges if, and only if, the integral \( \int_{\frac{1}{2} \pi}^{\infty} \frac{\cos x}{x^2} \, dx \) converges.

If \( x \geq 1 \) then \( \left| \frac{\cos x}{x^2} \right| \leq \frac{1}{x^2} \), and the integral \( \int_{1}^{\infty} \frac{dx}{x^2} \) is convergent \((p = 2 > 1 \text{ in the scale of powers at } \infty)\), so the comparison principle implies that \( \int_{\frac{1}{2} \pi}^{\infty} \frac{\cos x}{x^2} \, dx \) is convergent.

Therefore, the improper integral \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \) is convergent.
Remark on Example 8.

The comparison principle does not apply directly to Example 8, for

$$\int_{n\pi}^{(n+1)\pi} \left| \frac{\sin x}{x} \right| dx > \int_{(n+\frac{1}{6})\pi}^{(n+\frac{5}{6})\pi} \left| \frac{\sin x}{x} \right| dx > \frac{1}{4} \int_{n\pi}^{(n+1)\pi} \frac{dx}{x} \quad \text{if} \quad n \geq 1,$$

and therefore

$$\int_{\pi}^{n\pi} \left| \frac{\sin x}{x} \right| dx > \frac{1}{4} \int_{\pi}^{n\pi} \frac{dx}{x} = \frac{1}{4} \log n, \quad \text{if} \quad n \geq 2,$$

which implies that the improper integral

$$\int_{0}^{\infty} \left| \frac{\sin x}{x} \right| dx$$

is divergent.
You may wish to attempt the following exercises.

1. Show that the improper integral \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) is convergent.

   If you are interested, I can give you a list of bonus questions which lead to an evaluation of this integral using only techniques from this course.

2. Show that the improper integral \( \int_{0}^{\infty} \frac{\log(2\varphi)}{\varphi^2 + \varphi + 1} \, d\varphi \) is convergent.

   Then use symmetry to compute this integral.

3. Use symmetry to evaluate the improper integral \( \int_{0}^{\frac{1}{2}\pi} \log(\sin \vartheta) \, d\vartheta \).
Exercises, continued.

4. Determine all values of $p$ for which \[ \int_0^\infty \sin(x^p) \, dx \] is convergent.

5. If you are interested, I can give you a list of bonus questions which lead to an evaluation of the integral \[ \int_0^\infty \frac{\sin x}{x} \, dx, \] using only techniques from this course.