Often it is easy, using the product rule, to find a function whose derivative has a given function as a summand (replace a factor by one of its antiderivatives). In a number of significant cases, doing so constitutes decisive progress towards finding an antiderivative of the given function. *Partial integration*, or *integration by parts*, refers to any of various notations introduced to exploit such progress.

Example 1. To evaluate the integral

$$\int x^2 \cos x \, dx,$$

begin by integrating a factor (or *part*) of the integrand (leave the rest alone). There are two natural ways to integrate a part in this case, resulting in either $\frac{1}{3}x^3 \cos x$ or $x^2 \sin x$ (the first by integrating x^2, the second by integrating $\cos x$). The second result looks simpler; so choose it, and differentiate it. Now the integrand is a term of the resulting derivative (by the product rule),

$$\frac{d}{dx}\Big[x^2 \sin x\Big] = 2x \sin x + x^2 \cos x,$$

but so is the undesired term, or *remainder*, $2x \sin x$. The remainder is similar to, and simpler than, the original integrand, so it is reasonable to continue. Integrate the trigonometric factor (which yields $-2x \cos x$), but this time *change the sign* (which yields $2x \cos x$), since the goal now is to *cancel* (not match) the remainder. Finally, differentiate the result:

$$\frac{d}{dx}\Big[2x \cos x\Big] = 2 \cos x - 2x \sin x.$$

The new remainder (namely, $2 \cos x$) is easy to dispatch, for

$$\frac{d}{dx}\Big[-2 \sin x\Big] = -2 \cos x.$$

Summing these last three equations yields (by the sum rule for derivatives)

$$\frac{d}{dx}\Big[x^2 \sin x + 2x \cos x - 2 \sin x\Big] = x^2 \cos x,$$

or (as an integral equation, with the right hand side simplified a little)

$$\int x^2 \cos x \, dx = (x^2 - 2) \sin x + 2x \cos x + C,$$

as required. □

Such calculations will be organized[1] into a table. For example, the preceding calculations would be organized as follows.

$x^2 \sin x$	$\underline{x^2 \cos x}$	$+\ 2x \sin x$	
$2x \cos x$		$-\ 2x \sin x$	$+\ 2 \cos x$
$-2 \sin x$			$-\ 2 \cos x$

Aligning the terms in the right column (and underlining the integrand) provides inessential visual emphasis. Each row is populated as follows, beginning with a term in its right column.

> *Integrate a factor of the term (leave the other factor as is) and put the result in the left column, then differentiate the other factor of this result (leave the integrated factor as is) and add the new result to the right column.*

This insures that for any row (or any sum of multiples of rows), the right side is the derivative of the left side, and so the left side is an antiderivative of the right side.

Example 2. To evaluate

$$\int e^{3x} \sin 2x \, dx,$$

integrate the exponential factor at each stage.

$\frac{1}{3}e^{3x} \sin 2x$	$\underline{e^{3x} \sin 2x}$	$+\ \frac{2}{3}e^{3x} \cos 2x$
$-\frac{2}{9}e^{3x} \cos 2x$	$\frac{4}{9}\underline{e^{3x} \sin 2x}$	$-\ \frac{2}{3}e^{3x} \cos 2x$

The sum of the right column is $\frac{13}{9}$ of the integrand, and so

$$\begin{aligned}\int e^{3x} \sin 2x \, dx &= \tfrac{9}{13}\big\{\tfrac{1}{3}e^{3x} \sin 2x - \tfrac{2}{9}e^{3x} \cos 2x\big\} + C \\ &= \tfrac{1}{13}e^{3x}(3 \sin 2x - 2 \cos 2x) + C.\end{aligned}$$

□

Example 3. To evaluate[2]

$$\int x^3 e^{2x} \, dx,$$

integrate the exponential factor at each stage.

$\frac{1}{2}x^3 e^{2x}$	$\underline{x^3 e^{2x}}$	$+\ \frac{3}{2}x^2 e^{2x}$		
$-\frac{3}{4}x^2 e^{2x}$		$-\ \frac{3}{2}x^2 e^{2x}$	$-\ \frac{3}{2}x e^{2x}$	
$\frac{3}{4}x e^{2x}$			$\frac{3}{2}x e^{2x}$	$+\ \frac{3}{4}e^{2x}$
$-\frac{3}{8}e^{2x}$				$-\ \frac{3}{4}e^{2x}$

Writing the sum of the rows as an integral equation, and simplifying, yields

$$\int x^3 e^{2x} \, dx = \tfrac{1}{8}e^{2x}(4x^3 - 6x^2 + 6x - 3) + C.$$

□

Example 4. To evaluate[3]

$$\int x^2 (\ln x)^3 \, dx,$$

integrate the polynomial factor at each stage.

$\frac{1}{3}x^3 (\ln x)^3$	$\underline{x^2 (\ln x)^3}$	$+\ x^2 (\ln x)^2$		
$-\frac{1}{3}x^3 (\ln x)^2$		$-\ x^2 (\ln x)^2$	$-\ \frac{2}{3}x^2 \ln x$	
$\frac{2}{9}x^3 \ln x$			$\frac{2}{3}x^2 \ln x$	$+\ \frac{2}{9}x^2$
$-\frac{2}{27}x^3$				$-\ \frac{2}{9}x^2$

Summing and simplifying yields

$$\int x^2 (\ln x)^3 \, dx = \tfrac{1}{27}x^3\big\{9(\ln x)^3 - 9(\ln x)^2 + 6 \ln x - 2\big\} + C.$$

□

Example 5. To evaluate the integral

$$\int \frac{x^4 \, dx}{\sqrt{(2x^2+3)^5}},$$

integrate (a multiple of) x times a (negative) power of $2x^2 + 3$ at each stage.

$\dfrac{x^3}{(2x^2+3)^{3/2}}$	$-\ \dfrac{6x^4}{(2x^2+3)^{5/2}}$	$+\ \dfrac{3x^2}{(2x^2+3)^{3/2}}$	
$\dfrac{x}{(2x^2+3)^{1/2}}$		$-\ \dfrac{2x^2}{(2x^2+3)^{3/2}}$	$+\ \dfrac{1}{(2x^2+3)^{1/2}}$

The first row plus three halves of the second row expresses the integral equation

$$\frac{x^3}{\sqrt{(2x^2+3)^3}} + \frac{3x}{2\sqrt{2x^2+3}} = \int \frac{3\,dx}{2\sqrt{2x^2+3}} - \int \frac{6x^4\,dx}{\sqrt{(2x^2+3)^5}},$$

and so evaluating the first integral on the right, rearranging and simplifying, yields

$$\int \frac{x^4 \, dx}{\sqrt{(2x^2+3)^5}} = -\frac{x(8x^2+9)}{12\sqrt{(2x^2+3)^3}} + \tfrac{1}{8}\sqrt{2}\ln\big(x\sqrt{2} + \sqrt{2x^2+3}\big) + C.$$

□

Example 6. Integrating the factor $\sec^2\vartheta$ of $\sec^n\vartheta = \sec^{n-2}\vartheta \sec^2\vartheta$, and using $\tan^2\vartheta = \sec^2\vartheta - 1$ to expand the right column in powers of $\sec\vartheta$, yields

$\sec^{n-2}\vartheta \tan\vartheta$	$(n-1)\sec^n\vartheta - (n-2)\sec^{n-2}\vartheta.$

Rearranging the corresponding integral equation yields the reduction formula

$$\int \sec^n\vartheta \, d\vartheta = \frac{\sec^{n-2}\vartheta \tan\vartheta}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}\vartheta \, d\vartheta, \quad \text{if } n \neq 1.$$

□

The *partial integration formula* is just the product rule, $(fg)' = f'g + fg'$, written as an integral equation, either in symmetric form

$$fg + C = \int f'g + \int fg', \quad \text{or asymmetric form} \quad \int fg' = fg - \int f'g,$$

where at least one of f', g' is continuous. Two examples of the symmetric form are

$$x^2 \arcsin x + C = \int 2x \arcsin x \, dx + \int \frac{x^2 \, dx}{\sqrt{1-x^2}}, \quad \text{and}$$

$$\begin{aligned} x\sqrt{1-x^2} + C &= \int \sqrt{1-x^2}\, dx - \int \frac{x^2 \, dx}{\sqrt{1-x^2}} \\ &= \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{2x^2 \, dx}{\sqrt{1-x^2}}.\end{aligned}$$

Adding half the second to the first, evaluating, rearranging and simplifying, yields

$$\int x \arcsin x \, dx = \tfrac{1}{4}(2x^2 - 1) \arcsin x + \tfrac{1}{4}x\sqrt{1-x^2} + C.$$

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[1] after Leonard Gillman in *The College Mathematics Journal*, **22**, No. 5, 407–410.

[2] This is Example 1 (on Page 2) of `http://www.math.mcgill.ca/rags/JAC/Intparts.pdf`.

[3] This is Example 3 of `http://www2.johnabbott.qc.ca/ math/wb/nyb/turbo.pdf`.