THE RESOLUTION INTO PARTIAL FRACTIONS OF A RATIONAL EXPRESSION

§ 1. Introduction. — The problem of computing the indefinite integral

$$\int \frac{5x+1}{2x^3+x^2-2x-1}\,dx,$$

is resolved by the identity

$$\frac{5x+1}{2x^3+x^2-2x-1} = \frac{2}{2x+1} - \frac{2}{x+1} + \frac{1}{x-1}, \tag{1}$$

since the terms on the right are integrated easily by inspection. The right side of (1) is called the *resolution into partial fractions* (or the partial fractions decomposition) of the expression on the left.

§ 2. Background. — Every non-constant polynomial in one variable with real coefficients is a product of polynomials with real coefficients, each of which is either linear or quadratic. This is a formulation of the *fundamental theorem of algebra*, first proved (essentially) in the 1799 doctoral thesis of C. F. Gauß. The result is quite non-trivial. Besides including simple examples, such as

$$2x^3+x^2-2x-1 = (2x+1)(x+1)(x-1) \qquad\text{and}\qquad x^4+x^2+1 = (x^2-x+1)(x^2+x+1),$$

it predicts the existence of less apparent factorizations, such as

$$x^5-1 = (x-1)(x^2+\tau x+1)(x^2-\tau^{-1}x+1),$$

where $\tau = \frac{1}{2}+\frac{1}{2}\sqrt{5}$ is the golden section, and

$$x^3-3x-4 = \Big(x-\sqrt[3]{2+\sqrt{3}}-\sqrt[3]{2-\sqrt{3}}\Big)\Big(x^2+\big(\sqrt[3]{2+\sqrt{3}}+\sqrt[3]{2-\sqrt{3}}\big)x+\sqrt[3]{7+4\sqrt{3}}+\sqrt[3]{7-4\sqrt{3}}-1\Big),$$

and insures that

$$x^5-x-1$$

is a product one linear and two quadratic factors with real coefficients (some of which cannot be expressed in terms of integers using rational operations and radicals of any order).

§ 3. The resolution into partial fractions. — By § 2, the denominator of a rational expression in one variable with real coefficients is a product of powers, no two of which have a non-constant common factor, each of which is of the form $(ax+b)^m$, where $a \neq 0$ and m is a positive integer, or else $(\alpha x^2+\beta x+\gamma)^n$, where $\beta^2 < 4\alpha\gamma$ and n is a positive integer. The expression is the sum of a polynomial, a sum

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_m}{(ax+b)^m}$$

corresponding to each power of the first type, and a sum

$$\frac{B_1x+C_1}{\alpha x^2+\beta x+\gamma} + \frac{B_2x+C_2}{(\alpha x^2+\beta x+\gamma)^2} + \frac{B_3x+C_3}{(\alpha x^2+\beta x+\gamma)^3} + \cdots + \frac{B_nx+C_n}{(\alpha x^2+\beta x+\gamma)^n}$$

corresponding to each power of the second type; the polynomial and the coefficients (A_i, B_j, C_j) are determined uniquely and rationally by the expression and the factorization of its denominator. If the expression is reduced, then the last term in the the sum corresponding to each power is non-zero (in the displays, $A_m \neq 0$ and $B_n^2 + C_n^2 \neq 0$). The representation obtained is called the *resolution into partial fractions* (or the partial fractions decomposition) of the rational expression.

§ 4. Examples. — The examples below treat only the *form* of the resolution into partial fractions. Methods for computing the resolution are taken up later.

(1) Below, the numerator is quintic and the denominator is cubic, so the quotient is quadratic. Since $x^3-3x+2 = (x-1)^2(x+2)$, there are rational numbers a, b, c, A, B and C, such that

$$\frac{2x^5+x^4-7x^3-2}{x^3-3x+2} = ax^2+bx+c+\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}.$$

(2) The expression below is proper, so the quotient is zero. Since the denominator is factorized,

$$\frac{15x+7}{(3x-2)^2(x^2+2x+2)} = \frac{a}{3x-2} + \frac{b}{(3x-2)^2} + \frac{cx+d}{x^2+2x+2},$$

for certain rational numbers a, b, c and d.

(3) The expression below is proper, so the quotient is zero. Since $x^3+8 = (x+2)(x^2-2x+4)$, the denominator is $(x+1)^3(x+2)^2(x^2-2x+4)^2$, and so

$$\frac{6x^2-2x+3}{(x^3+8)^2(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x+2} + \frac{E}{(x+2)^2} + \frac{Fx+G}{x^2-2x+4} + \frac{Hx+I}{(x^2-2x+4)^2},$$

for certain rational numbers A, B, C, D, E, F, G, H and I.

(4) In the expression below, cancelling the common factor $x-1$ gives

$$\frac{x^6-1}{x^5-1} = \frac{x^5+x^4+x^3+x^2+x+1}{x^4+x^3+x^2+x+1}.$$

The quotient is linear and, via the third factorization from § 2, there are rational numbers a and b, and real (algebraic) numbers α, β, γ and δ such that

$$\frac{x^5+x^4+x^3+x^2+x+1}{x^4+x^3+x^2+x+1} = ax+b+\frac{\alpha x+\beta}{x^2+(\frac{1}{2}+\frac{1}{2}\sqrt{5})x+1} + \frac{\gamma x+\delta}{x^2+(\frac{1}{2}-\frac{1}{2}\sqrt{5})x+1}.$$

Remark. — It is a simple matter to integrate a rational expression with a quadratic denominator so, for the purposes of integration, nothing is lost by leaving unresolved an expression such as

$$\frac{2x+1}{x^2-2x-2},$$

although strictly, its resolution into partial fractions (over the real numbers) is

$$\frac{2+\sqrt{3}}{2(x-1-\sqrt{3})} + \frac{2-\sqrt{3}}{2(x-1+\sqrt{3})}.$$

Likewise, for the purpose of integration, since it is clear by inspection that

$$\int \frac{5x^4-1}{x^5-x-1}\,dx = \log|x^5-x-1|,$$

nothing is gained by (the rather formidable task of) resolving the integrand into partial fractions.

§ 5. Computing coefficients using elimination. — The coefficients in the resolution into partial fractions of a rational expression are the unknowns of a system of linear equations which can be solved by *elimination* (adding multiples of an equation to other equations so as to eliminate one unknown from a system of fewer equations, *&c.*). As an example, consider the rational expression

$$\frac{2x^3+11x-25}{4x^4+16x^2+25} = \frac{2x^3+11x-25}{(2x^2-2x+5)(2x^2+2x+5)}.$$

There are unique rational numbers a, b, c and d, such that

$$\frac{ax+b}{2x^2-2x+5} + \frac{cx+d}{2x^2+2x+5} = \frac{2x^3+11x-25}{4x^4+16x^2+25},$$

or (clearing denominators)

$$(ax+b)(2x^2+2x+5)+(cx+d)(2x^2-2x+5) = 2x^3+11x-25.$$

Comparing the cubic, quadratic, linear and constant coefficients yields, respectively,

$$a+c=1, \qquad a+b-c+d=0, \qquad 5a+2b+5c-2d=11 \qquad\text{and}\qquad b+d=-5$$

(after simplification). Subtracting the first equation from the second equation, and subtracting five times the first equation from the third equation (and multiplying the result by $\frac{1}{2}$), gives

$$b-2c+d=-1 \qquad\text{and}\qquad b-d=3.$$

Subtracting the fourth equation from the fifth and sixth equations gives $-2c=4$ and $-2d=8$, so $c=-2$ and $d=-4$. The first and fourth equations then give $a=3$ and $b=-1$. Therefore,

$$\frac{2x^3+11x-25}{4x^4+16x^2+25} = \frac{3x-1}{2x^2-2x+5} - \frac{2x+4}{2x^2+2x+5}.$$

The integration of these partial fractions is taken up next. Since $3x-1=\frac{3}{2}(2x-1)+\frac{1}{2}$ and $2(2x^2-2x+5)=(2x-1)^2+9$, it follows that

$$\int\frac{3x-1}{2x^2-2x+5}\,dx=\frac{3}{2}\int\frac{2x-1}{2x^2-2x+5}\,dx+\int\frac{dx}{(2x-1)^2+9}$$
$$=\tfrac{3}{4}\log(2x^2-2x+5)+\tfrac{1}{6}\arctan\left(\tfrac{1}{3}(2x-1)\right).$$

Likewise, $2x+4=2x+1+3$ and $2(2x^2+2x+5)=(2x+1)^2+9$, and hence

$$\int\frac{2x+4}{2x^2+2x+5}\,dx=\int\frac{2x+1}{2x^2+2x+5}\,dx+6\int\frac{dx}{(2x+1)^2+9}$$
$$=\tfrac{1}{2}\log(2x^2+2x+5)+\arctan\left(\tfrac{1}{3}(2x+1)\right).$$

Combining these results gives

$$\int\frac{4x^3+11x-25}{4x^4+16x^2+25}\,dx=\frac{1}{4}\log\frac{(2x^2-2x+5)^3}{(2x^2+2x+5)^2}+\frac{1}{6}\arctan\left(\tfrac{1}{3}(2x-1)\right)-\arctan\left(\tfrac{1}{3}(2x+1)\right).$$

§ 6. Linear factors. — If the denominator of a rational expression has linear factors, then certain coefficients in its resolution into partial fractions are apparent values of simply described rational expressions (for a rational equation in one variable with many—*e.g.*, infinitely many—solutions is an identity). This is amply illustrated by the first example of § 1. Since

$$2x^3+x^2-2x-1=(2x+1)(x+1)(x-1),$$

there are rational numbers A, B and C such that

$$\frac{A}{2x+1}+\frac{B}{x+1}+\frac{C}{x-1}=\frac{5x+1}{(2x+1)(x+1)(x-1)}.$$

Multiplying by $x-1$ gives

$$\frac{A(x-1)}{2x+1}+\frac{B(x-1)}{x+1}+C=\frac{5x+1}{(2x+1)(x+1)}.$$

The first two terms on the left side of this last equation vanish if $x=1$, so

$$C=\left.\frac{5x+1}{(2x+1)(x+1)}\right|_{x=1}=\frac{5\cdot1+1}{(2\cdot1+1)(1+1)}=1.$$

By the same reasoning,

$$A=\left.\frac{5x+1}{(x+1)(x-1)}\right|_{x=-\frac{1}{2}}=2,\qquad\text{and}\qquad B=\left.\frac{5x+1}{(2x+1)(x-1)}\right|_{x=-1}=-2.$$

Therefore,

$$\int\frac{5x+1}{2x^3+x^2-2x-1}\,dx=\int\left\{\frac{2}{2x+1}-\frac{2}{x+1}+\frac{1}{x-1}\right\}dx=\log\left|\frac{(2x+1)(x-1)}{(x+1)^2}\right|.$$

Next, consider the first example of § 4. Division yields

$$\frac{2x^5+x^4-7x^3-2}{x^3-3x+2}=2x^2+x-1+\frac{-x^2-5x}{(x+2)(x-1)^2}.$$

The resolution into partial fractions of the proper part is

$$\frac{-x^2-5x}{(x+2)(x-1)^2}=\frac{2}{3(x+2)}-\frac{5}{3(x-1)}-\frac{2}{(x-1)^2},$$

where the first and last coefficients are computed as above giving, respectively,

$$\left.\frac{-x^2-5x}{(x-1)^2}\right|_{x=-2}=\frac{2}{3}\qquad\text{and}\qquad\left.\frac{-x^2-5x}{x+2}\right|_{x=1}=-2.$$

Multiplying by $x-1$ does not isolate the second coefficient in any significant manner. However, the second coefficient is $-\frac{5}{3}$ by inspection, since $-\frac{5}{3}+\frac{2}{3}=-1$ is the quadratic coefficient on the left.

Hence,

$$\int\frac{2x^5+x^4-7x^3-2}{x^3-3x+2}\,dx=\int\left\{2x^2+x-1+\frac{2}{3(x+2)}-\frac{5}{3(x-1)}-\frac{2}{(x-1)^2}\right\}dx$$
$$=\tfrac{2}{3}x^3+\tfrac{1}{2}x^2-x+\frac{2}{x-1}+\frac{1}{3}\log\left|\frac{(x+2)^2}{(x-1)^5}\right|.$$

§ 7. An exercise. — Once the techniques are understood, it will be easy to compute the integral

$$\int\frac{15x+7}{(3x-2)^2(x^2+2x+2)}\,dx$$

in about ten minutes. [The result is $-\frac{3}{2}(3x-2)^{-1}-\frac{1}{2}\arctan(x+1)$.]

§ 8. Ostrogradsky's manœuvre. — If the denominator of an integrand has repeated quadratic factors, the process can be streamlined because the form of the result is predictable. This will be illustrated by an example. Since $x^3-1=(x-1)(x^2+x+1)$, there are rational numbers a, b, c, A, B and C such that

$$\int\frac{dx}{(x^3-1)^2}=\frac{ax^2+bx+c}{x^3-1}+\int\left\{\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\right\}dx.$$

Differentiating and clearing denominators gives

$$1=(2ax+b)(x^3-1)-3x^2(ax^2+bx+c)+A(x^2+x+1)(x^3-1)+(Bx+C)(x-1)(x^3-1).$$

Comparing the coefficients of x^5, x^4, x^3, x^2 and x, and the constant coefficient, gives, respectively,

$$A+B=0,\qquad -a+A-B+C=0,\qquad -2b+A-C=0,$$
$$-3c-A-B=0,\qquad -2a-A+B-C=0\quad\text{and}\quad -b-A+C=1.$$

Adding these equations in columns gives $c=0$, $a=0$ and $b=-\frac{1}{3}$. Adding the first equation to the second equation, and revising the third equation, gives

$$2A+C=0\quad\text{and}\quad A-C=-\tfrac{2}{3},\quad\text{so}\quad A=-\tfrac{2}{9},\quad\text{and}\quad C=\tfrac{4}{9}.$$

The first equation now gives $B=\frac{2}{9}$. Therefore,

$$\int\frac{dx}{(x^3-1)^2}=-\frac{x}{3(x^3-1)}+\frac{1}{9}\int\left\{\frac{-2}{x-1}+\frac{2x+4}{x^2+x+1}\right\}dx.$$

That $2x+4=2x+1+3$, and $4(x^2+x+1)=(2x+1)^2+3$ gives (as in the calculations at the end of § 5)

$$\int\frac{2x+4}{x^2+x+1}\,dx=\log(x^2+x+1)+2\sqrt{3}\arctan\left(\tfrac{1}{3}(2x-1)\sqrt{3}\right).$$

Combining these results gives

$$\int\frac{dx}{(x^3-1)^2}=-\frac{x}{3(x^3-1)}+\frac{1}{9}\log\frac{x^2+x+1}{(x-1)^2}+\frac{2}{9}\sqrt{3}\arctan\left(\tfrac{1}{3}(2x-1)\sqrt{3}\right).$$

The reader should compare the work involved in this computation with that of a direct approach.

§ 9. Reasons. — The next paragraph gives a brief explanation of the existence and uniqueness of the resolution into partial fractions of a rational expression in one variable with real coefficients.

If non-constant polynomials f, g have no common factor, repeated division yields polynomials a, b such that $af+bg=1$, so a rational expression with denominator fg is the sum of a rational expression with denominator f and a rational expression with denominator g. By the fundamental theorem of algebra, every rational expression is a sum of rational expressions p/q^n, where $n\geqslant 0$ and q is linear or quadratic, and repeated division (by q) yields a resolution into partial fractions. The polynomial in a resolution is unique by division. The uniqueness of the coefficients over powers of linear polynomials is a consequence of § 6. The uniqueness of the remaining coefficients is a consequence of the fact that if $p,q_1,\ldots,q_k$ are quadratic polynomials with negative discriminant, none of which is divisible by any other, $\ell_1,\ldots,\ell_k$ are positive integers, a, b are real numbers, and the product $(ax+b)q_1^{\ell_1}\cdots q_k^{\ell_k}$ is divisible by p, then $a=0$ and $b=0$ (which follows from division).

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