In most cases an integral is evaluated using an inverse trigonometric substitution, and in every case by a more efficient method. Where an inverse trigonometric substitution is used it is given, and it is left to the reader to draw the appropriate triangle, *etc.*

1. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$ The change of variable $x = 2\sin\vartheta$, yields

$$\int \frac{(2\cos\vartheta)(2\cos\vartheta\,d\vartheta)}{2\sin\vartheta} = 2\int \frac{1-\sin^2\vartheta}{\sin\vartheta} = 2\int(\csc\vartheta-\sin\vartheta)\,d\vartheta$$
$$= -2\ln|\csc\vartheta+\cot\vartheta| + 2\cos\vartheta + C$$
$$= -2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C.$$

Alternatively, the change of variable $t^2 = 4-x^2$ yields

$$\int \frac{t^2}{t^2-4}\,dt = \int\left\{1+\frac{4}{t^2-4}\right\}dt = t + \ln\left|\frac{t-2}{t+2}\right| + C$$
$$= \sqrt{4-x^2} + \ln\left|\frac{\sqrt{4-x^2}-2}{\sqrt{4-x^2}+2}\right| + C = \sqrt{4-x^2} + 2\ln\left|\frac{\sqrt{4-x^2}-2}{x}\right| + C.$$

2. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-4}}$ The change of variable $x = 2\sec\vartheta$ yields

$$\int \frac{(2\sec\vartheta)(2\sec\vartheta\tan\vartheta\,d\vartheta)}{2\tan\vartheta} = 2\int\sec^2\vartheta\,d\vartheta = \tan\vartheta + C = \sqrt{x^2-4}+C.$$

Alternatively, the change of variable $t^2 = x^2-4$ yields $\displaystyle\int dt = t + C = \sqrt{x^2-4}+C$.

3. $\displaystyle\int \frac{x+3}{x^2+2x+5}\,dx = \int\frac{(x+1)+2}{(x+1)^2+4}\,dx.$ Letting $x+1 = 2\tan\vartheta$ yields

$$\int \frac{(2\tan\vartheta+2)(2\sec^2\vartheta\,d\vartheta)}{4\sec^2\vartheta} = \int(\tan\vartheta+1)\,d\vartheta = \vartheta + \ln|\sec\vartheta| + C$$
$$= \arctan\tfrac12(x+1) + \ln\sqrt{x^2+2x+5} + C.$$

Alternatively, splitting the integral yields

$$\int\frac{(x+1)\,dx}{x^2+2x+5} + \int\frac{2\,dx}{(x+1)^2+4} = \tfrac12\ln(x^2+2x+5) + \arctan\tfrac12(x+1) + C.$$

4. This integral involves a direct application of a standard integral formula.

$$\int\frac{dx}{\sqrt{6-4x-2x^2}} = \frac{1}{\sqrt2}\int\frac{dx}{\sqrt{4-(x+1)^2}} = \tfrac12\sqrt2\arcsin\tfrac12(x+1) + C.$$

5. $\displaystyle\int\sqrt{9-8x^2}\,dx$ The change of variable $(2\sqrt2)x = 3\sin\vartheta$ yields

$$\int(3\cos\vartheta)\left(\tfrac34\sqrt2\cos\vartheta\,d\vartheta\right) = \tfrac94\sqrt2\int\cos^2\vartheta\,d\vartheta$$
$$= \tfrac98\sqrt2(\vartheta+\sin\vartheta\cos\vartheta) + C$$
$$= \tfrac98\sqrt2\arcsin\left(\tfrac23x\sqrt2\right) + \tfrac12x\sqrt{9-8x^2} + C.$$

Alternatively, applying the reduction formula

$$\int(\alpha t^2+\beta)^\nu\,dt = \frac{t(\alpha t^2+\beta)^\nu}{2\nu+1} + \frac{2\nu\beta}{2\nu+1}\int(\alpha t^2+\beta)^{\nu-1}\,dt,$$

yields

$$\tfrac12x\sqrt{9-8x^2} + \tfrac92\int\frac{dx}{\sqrt{9-8x^2}} = \tfrac12x\sqrt{9-8x^2} + \tfrac98\sqrt2\arcsin\left(\tfrac23x\sqrt2\right) + C.$$

Another approach is to use undetermined coefficients. Rationalize the numerator, and find A, B and α such that

$$\int\frac{9-8x^2}{\sqrt{9-8x^2}}\,dx = (Ax+B)\sqrt{9-8x^2} + \int\frac{\alpha\,dx}{\sqrt{9-8x^2}}.$$

Differentiating this equation, clearing denominators and collecting like terms, yields $9-8x^2 = -16Ax^2 - 8Bx + 9A + \alpha$, and so $A = \frac12$, $B = 0$ and $\alpha = \frac92$, as above.

6. $\displaystyle\int(a^2-x^2)^{3/2}\,dx$ The change of variable $x = a\sin\vartheta$ yields

$$\int(a^3\cos^3\vartheta)(a\cos\vartheta\,d\vartheta) = a^4\int\cos^4\vartheta\,d\vartheta$$
$$= \tfrac14a^4\int(1+2\cos2\vartheta+\cos^2 2\vartheta)\,d\vartheta$$
$$= \tfrac14a^4\left\{\vartheta+\sin2\vartheta+\tfrac14(2\vartheta+\sin2\vartheta\cos2\vartheta)\right\} + C$$
$$= \tfrac14a^4\left\{\tfrac32\vartheta+\tfrac14\sin2\vartheta(\cos2\vartheta+4)\right\} + C$$
$$= \tfrac18a^4\left\{3\vartheta+\sin\vartheta\cos\vartheta(5-2\sin^2\vartheta)\right\} + C$$
$$= \tfrac18a^4\left\{3\arcsin(x/a) + (x/a^2)\sqrt{a^2-x^2}\left(5-2(x^2/a^2)\right)\right\} + C$$
$$= \tfrac38a^4\arcsin(x/a) + \tfrac18x(5a^2-2x^2)\sqrt{a^2-x^2} + C.$$

Alternatively, the reduction formula given in the previous problem yields

$$\int(a^2-x^2)^{3/2}\,dx = \tfrac14x(a^2-x^2)^{3/2} + \tfrac34a^2\int\sqrt{a^2-x^2}\,dx$$
$$= \tfrac14x(a^2-x^2)^{3/2} + \tfrac38a^2x\sqrt{a^2-x^2} + \tfrac38a^4\int\frac{dx}{\sqrt{a^2-x^2}}$$
$$= \tfrac14x(a^2-x^2)^{3/2} + \tfrac38a^2x\sqrt{a^2-x^2} + \tfrac38a^4\arcsin(x/a) + C$$
$$= \tfrac18x(5a^2-2x^2)\sqrt{a^2-x^2} + \tfrac38a^4\arcsin(x/a) + C.$$

7. $\displaystyle\int\frac{dx}{\sqrt{x^2-9}} = \ln\left|x+\sqrt{x^2-9}\right| + C$, by a standard integral formula.

Alternatively, the change of variable $x = 3\sec\vartheta$ yields

$$\int\frac{3\sec\vartheta\tan\vartheta\,d\vartheta}{3\tan\vartheta} = \int\sec\vartheta\,d\vartheta = \ln|\sec\vartheta+\tan\vartheta| + C$$
$$= \ln\left|x+\sqrt{x^2-9}\right| + C.$$

8. This integral involves a direct application of a standard integral formula.

$$\int\frac{dx}{\sqrt{4x-4x^2}} = \int\frac{dx}{\sqrt{1-(2x-1)^2}} = \tfrac12\arcsin(2x-1) + C.$$

9. $\displaystyle\int\frac{\sqrt{x^2-9}}{x}\,dx$ The change of variable $x = 3\sec\vartheta$ yields

$$\int\frac{(3\tan\vartheta)(3\sec\vartheta\tan\vartheta\,d\vartheta)}{3\sec\vartheta} = 3\int\tan^2\vartheta\,d\vartheta = 3(\tan\vartheta-\vartheta) + C$$
$$= \sqrt{x^2-9} - 3\,\mathrm{arcsec}\,\tfrac13x + C.$$

Alternatively, the change of variable $t^2 = x^2-9$ yields

$$\int\frac{t^2\,dt}{t^2+9} = \int\left\{1-\frac{9}{t^2+9}\right\}dt = t - 3\arctan\tfrac13t + C$$
$$= \sqrt{x^2-9} - 3\arctan\tfrac13\sqrt{x^2-9} + C.$$

10. $\displaystyle\int\frac{dx}{\sqrt{x^2+6x+13}} = \int\frac{dx}{\sqrt{(x+3)^2+4}} = \ln\left(x+3+\sqrt{x^2+6x+13}\right) + C,$ by a standard integral formula.

Alternatively, the change of variable $x+3 = 4\tan\vartheta$ yields

$$\int\frac{2\sec\vartheta\tan\vartheta\,d\vartheta}{2\tan\vartheta} = \int\sec\vartheta\,d\vartheta = \ln|\sec\vartheta+\tan\vartheta| + C$$
$$= \ln\left(x+3+\sqrt{x^2+6x+13}\right) + C.$$

11. $\displaystyle\int\frac{x^4+1}{x^2+1}\,dx$ The change of variable $x = \tan\vartheta$ yields

$$\int\frac{(\tan^4\vartheta+1)(\sec^2\vartheta\,d\vartheta)}{\sec^2+1} = \int(\tan^4\vartheta+1)\,d\vartheta$$
$$= \int\left\{(\sec^2\vartheta-1)\tan^2\vartheta+1\right\}d\vartheta = \int(\tan^2\vartheta\sec^2\vartheta-\sec^2\vartheta+2)\,d\vartheta$$
$$= \tfrac13\tan^3\vartheta - \tan\vartheta + 2\vartheta + C = \tfrac13x^3 - x + 2\arctan x + C.$$

Alternatively, division yields

$$\int\left\{x^2-1+\frac{2}{x^2+1}\right\}dx = \tfrac13x^3 - x + 2\arctan x + C.$$

12. This integral will be evaluated using undetermined coefficients. There are real numbers A, B, C and α such that

$$\mathscr{I} = \int\frac{x^3\,dx}{(6x^2+12x-5)^{3/2}} = \frac{Ax^2+Bx+C}{\sqrt{6x^2+12x-5}} + \int\frac{\alpha}{\sqrt{6x^2+12x-5}}\,dx.$$

Differentiating, clearing denominators, and collecting like terms on the right hand side yields

$$x^3 = 6Ax^3 + (18A+6\alpha)x^2 + (-10A+6B-6C+12\alpha)x - 5B - 6C - 5\alpha.$$

Equating coefficients yields $A = \frac16$, $\alpha = -\frac12$, $6B-6C = \frac{23}{3}$, $5B+6C = \frac52$ (the values of A and α have been used to simplify the last two equations). Adding the last two equations and solving for B gives $B = \frac{61}{66}$, and hence $C = -\frac{35}{99}$ (using third or fourth equation). Therefore,

$$\mathscr{I} = \frac{(1/6)x^2+(61/66)x-35/99}{\sqrt{6x^2+12x-5}} - \tfrac12\int\frac{dx}{\sqrt{\left((x+1)\sqrt6\right)^2-11}}$$
$$= \frac{33x^2+183x-70}{198\sqrt{6x^2+12x-5}} - \tfrac1{12}\sqrt6\ln\left|(x+1)\sqrt6+\sqrt{6x^2+12x-5}\right| + C.$$

To evaluate this integral using an inverse trigonometric substitution would be a huge waste of time.

Trig sub is a method of choice for the same reason that three-legged races are popular.

$\mathscr{W}\beta$