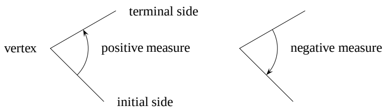
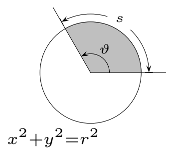
1. **Review of angles and associated concepts.** An angle has a *vertex*, an *initial side* and a *terminal side*, as illustrated below. It is in *standard position* if its initial side is the positive x-axis. The radian *measure* of an angle is the number of revolutions of the angle times 2π, with counterclockwise revolutions positive and clockwise revolutions negative. Throughout this note an angle is identified with its radian measure and angles are in standard position. On a circle of positive radius r centred at the vertex of an angle ϑ, the arc traced by ϑ has length $r\vartheta$ and the sector swept out by ϑ has area $\frac{1}{2}r^2\vartheta$, where $0 \leqslant \vartheta \leqslant 2\pi$ (and these quantities extend naturally to all real values of ϑ).





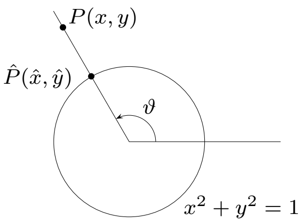
$\frac{1}{3}$ revolution (in standard position):

Angle: $\vartheta = \frac{1}{3} \cdot 2\pi = \frac{2}{3}\pi$

Arc length: $s = r\vartheta = \frac{2}{3}\pi r$

Area of the sector: $A = \frac{1}{2}r^2\vartheta = \frac{1}{3}\pi r^2$

2. **The sine, cosine and tangent of an angle.** Given an angle ϑ, let $P(x, y)$ be a point on its terminal side (different from the origin). The terminal side of ϑ meets the unit circle $x^2 + y^2 = 1$ at the point $\hat{P}(\hat{x}, \hat{y})$, where $\hat{x} = x/r$, $\hat{y} = y/r$ and $r = \sqrt{x^2 + y^2}$ is the distance between P and the origin. The sine, cosine and tangent of ϑ are defined as follows.



$$\cos\vartheta = \frac{x}{r} = \hat{x}$$

$$\sin\vartheta = \frac{y}{r} = \hat{y} \qquad \text{where } r = \sqrt{x^2 + y^2}$$

$$\tan\vartheta = \frac{y}{x} = \frac{\hat{y}}{\hat{x}} = \frac{\sin\vartheta}{\cos\vartheta} \qquad \text{provided } x \neq 0$$

Note: $\cos^2\vartheta + \sin^2\vartheta = 1$, since $\cos\vartheta$ and $\sin\vartheta$ are the coordinates of a point on the unit circle $x^2+y^2 = 1$. For the same reason, $-1 \leqslant \cos\vartheta \leqslant 1$ and $-1 \leqslant \sin\vartheta \leqslant 1$.

Any point on the terminal side of ϑ (different from the origin) can be used to compute these functions of ϑ (by similarity of triangles). Such computations usually involve (explicitly or implicitly) Pythagoras' formula.

For example, if $P(-3, 4)$ is on the terminal side of ϑ then $x = -3$, $y = 4$ and $r = \sqrt{(-3)^2 + 4^2} = 5$; so $\cos\vartheta = -\frac{3}{5}$, $\sin\vartheta = \frac{4}{5}$ and $\tan\vartheta = -\frac{4}{3}$.

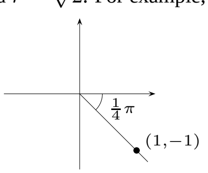
3. **Values at quadrant angles.** The terminal side of a *quadrant* angle is a positive or negative axis.

3.1. *Horizontal angles.* The terminal side of ϑ lies on the x-axis if $\vartheta = m\pi$, where m is an integer. Then $\sin\vartheta = \tan\vartheta = 0$. If m is even then $\cos\vartheta = 1$ (since $(1, 0)$ is on the terminal side of ϑ) and if m is odd then $\cos\vartheta = -1$ (since $(-1, 0)$ is on the terminal side of ϑ).

3.2. *Vertical angles.* The terminal side of ϑ lies on the y-axis if $\vartheta = \frac{1}{2}n\pi$, where n is an odd integer. Then $\cos\vartheta = 0$ and $\tan\vartheta$ is undefined (and these are the only angles whose tangent is undefined). If n is one more than a multiple of 4 then $\sin\vartheta = 1$ (since $(0, 1)$ lies on the terminal side of ϑ), and if n is one less than a multiple of 4 then $\sin\vartheta = -1$ (since $(0, -1)$ lies on the terminal side of ϑ).

4. **Values at other geometrically simple angles.** There are a few other cases in which finding a point on the terminal side of an angle can be done by very simple geometry.

4.1. *Halving a quadrant.* A point on the terminal side of an odd multiple of $\frac{1}{4}\pi$ lies on the line $y = x$, or on the line $y = -x$, so one can take $x = \pm 1$, $y = \pm 1$ (depending on which quadrant contains the terminal side), and $r = \sqrt{2}$. For example, since $\frac{7}{4}\pi = 2\pi - \frac{1}{4}\pi$, $(1, -1)$ lies on the terminal side of $\frac{7}{4}\pi$.



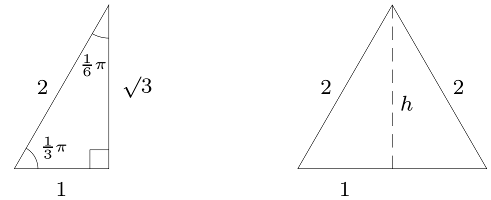
So $x = 1$, $y = -1$ and $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$; therefore,

$\cos\frac{7}{4}\pi = 1/\sqrt{2} = \frac{1}{2}\sqrt{2}$,

$\sin\frac{7}{4}\pi = (-1)/\sqrt{2} = -\frac{1}{2}\sqrt{2}$, and

$\tan\frac{7}{4}\pi = (-1)/1 = -1$.

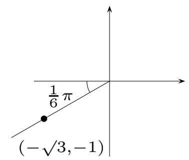
4.2. *Halving an equilateral triangle.* A point on the terminal side of a (reduced) rational multiple of π with denominator 3 or 6 can be found by dividing an equilateral triangle in half with any of its angle bisectors. Since the angles in the triangle sum to π, they are each $\frac{1}{3}\pi$, and bisecting any one produces $\frac{1}{6}\pi$. Below the side length 2 was chosen for arithmetical convenience only (in principle, any side length would do).



$1^2 + h^2 = 2^2$

$\therefore\ h = \sqrt{3}$

For example, since $\frac{7}{6}\pi = \pi + \frac{1}{6}\pi$, $(-\sqrt{3}, -1)$ lies on the terminal side of $\frac{7}{6}\pi$.



So $x = -\sqrt{3}$, $y = -1$ and $r = 2$; therefore,

$\cos\frac{7}{6}\pi = (-\sqrt{3})/2 = -\frac{1}{2}\sqrt{3}$,

$\sin\frac{7}{6}\pi = (-1)/2 = -\frac{1}{2}$, and

$\tan\frac{7}{6}\pi = (-1)/(-\sqrt{3}) = \frac{1}{3}\sqrt{3}$.

So it is not difficult to calculate trigonometric functions of $\frac{p}{q}\pi$, where $\frac{p}{q}$ is reduced and $q = 1, 2, 3, 4$ or 6.

5. **Symmetries, I: translations.** A few trigonometric identities are seen by imagining a point $P(x, y)$ on the terminal side of ϑ, and the point $\bar{P}$ that results from a translation (or rotation) through a quadrant angle. (*Exercise:* Illustrate each group of identities by drawing and labelling a suitable picture.)

5.1. *Translation by a full revolution.* The point $P(x, y)$ is also on the terminal side of $\vartheta + 2\pi$; so

$$\sin(\vartheta + 2\pi) = \sin\vartheta \qquad \text{and} \qquad \cos(\vartheta + 2\pi) = \cos\vartheta.$$

These identities are sometimes expressed by saying that *the sine and cosine functions have period* 2π.

5.2. *Translation by a half revolution.* The point $\bar{P}(-x, -y)$ is on the terminal side of $\vartheta + \pi$; so

$$\cos(\vartheta + \pi) = -\cos\vartheta, \qquad \sin(\vartheta + \pi) = -\sin\vartheta \qquad \text{and} \qquad \tan(\vartheta + \pi) = \tan\vartheta.$$

The last identity is sometimes expressed by saying that *the tangent function has period* π.

5.3. *Translation by a quarter revolution.* The point $\bar{P}(-y, x)$ is on the terminal side of $\vartheta + \frac{1}{2}\pi$; so

$$\cos(\vartheta + \tfrac{1}{2}\pi) = -\sin\vartheta \qquad \text{and} \qquad \sin(\vartheta + \tfrac{1}{2}\pi) = \cos\vartheta.$$

6. **Symmetries, II: reflections.** A few other trigonometric identities are seen by imagining a point $P(x, y)$ on the terminal side of ϑ, and the point $\bar{P}$ that results from a reflection, in an axis, or in the line $y = x$. (*Exercise:* Illustrate each group of identities by drawing and labelling a suitable picture.)

6.1. *Reflection in the x-axis.* The point $\bar{P}(x, -y)$ is on the terminal side of $-\vartheta$; so

$$\cos(-\vartheta) = \cos\vartheta, \qquad \sin(-\vartheta) = -\sin\vartheta \qquad \text{and} \qquad \tan(-\vartheta) = -\tan\vartheta.$$

These are often expressed by saying that *cosine is an even function, and sine and tangent are odd functions*.

6.2. *Reflection in the y-axis.* The point $\bar{P}(-x, y)$ is on the terminal side of $\pi - \vartheta$; so

$$\cos(\pi - \vartheta) = -\cos\vartheta \qquad \text{and} \qquad \sin(\pi - \vartheta) = \sin\vartheta.$$

These are sometimes called *supplementary identities*, because ϑ and $\pi - \vartheta$ are sometimes called *supplements*.

6.3. *Reflection in the line $y = x$.* The point $\bar{P}(y, x)$ is on the terminal side of $\frac{1}{2}\pi - \vartheta$; so

$$\cos(\tfrac{1}{2}\pi - \vartheta) = \sin\vartheta \qquad \text{and} \qquad \sin(\tfrac{1}{2}\pi - \vartheta) = \cos\vartheta.$$

These are sometimes called *complementary identities*, or *cofunction identities*, because ϑ and $\frac{1}{2}\pi - \vartheta$ are sometimes called *complements*.