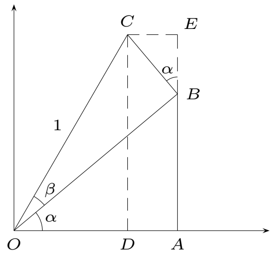
1. **Sum and difference identities.** The display below indicates a derivation of the sum identities for the sine and cosine functions, provided that α, β and $\alpha+\beta$ are acute.



$$\begin{aligned}\cos(\alpha+\beta) &= |OD| = |OA| - |DA| = |OA| - |CE| \\ &= (\cos\alpha)|OB| - (\sin\alpha)|BC| \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\,, \quad \text{and}\end{aligned}$$

$$\begin{aligned}\sin(\alpha+\beta) &= |DC| = |AE| = |AB| + |BE| \\ &= (\sin\alpha)|OB| + (\cos\alpha)|BC| \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\,.\end{aligned}$$

The derivation can be modified to cover the case when $\alpha+\beta$ is obtuse, and then extended to all real values of α and β using the symmetries at the end of *Basic trigonometry I*. As doing so is tedious, it is worth mentioning a general principle from which the identity for all real values of α and β follows from the special case displayed above: *If two rational functions of sines and cosines are equal on an interval of positive length, then they are equal on their common domain.* The general phenomenon underlying this fact is called analytic continuation.

Replacing β by $-\beta$ in the sum identities, and using that $\cos(-\beta)=\cos\beta$ and $\sin(-\beta)=-\sin\beta$ (see the symmetry 6.1 in *Basic trigonometry I*), yields the difference identities for the sine and cosine functions, which are given along with the sum identities below.

$$\begin{aligned}\cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ \sin(\alpha\pm\beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\end{aligned}$$

Dividing each identity for the sine function by the corresponding identity for the cosine function, and then expressing the right hand side of the result in terms of the tangent function (multiplying and dividing by the reciprocal of $\cos\alpha\cos\beta$), yields the sum and difference identities for the tangent function.

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

2. **Double angle identities.** Letting $\alpha=\beta=\vartheta$ in the sum identities results in double angle identities, which express a trigonometric function of 2ϑ in terms of trigonometric functions of ϑ.

$$\sin 2\vartheta = 2\sin\vartheta\cos\vartheta \qquad \cos 2\vartheta = \cos^2\vartheta - \sin^2\vartheta \qquad \tan 2\vartheta = \frac{2\tan\vartheta}{1-\tan^2\vartheta}.$$

Using the (Pythagorean) identity $\cos^2\vartheta+\sin^2\vartheta=1$ to eliminate $\sin^2\vartheta$, or $\cos^2\vartheta$, from the double angle identity for the cosine, gives two new forms of this identity,

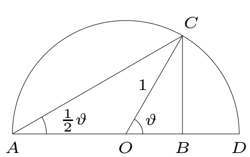
$$\cos 2\vartheta = 2\cos^2\vartheta - 1 \qquad \text{and} \qquad \cos 2\vartheta = 1-2\sin^2\vartheta,$$

which are frequently useful.

3. **Half angle identities.** Solving the last two identities for $\cos^2\vartheta$, respectively $\sin^2\vartheta$, and then replacing ϑ by $\frac12\vartheta$, gives the half angle identities,

$$\cos^2\tfrac12\vartheta = \tfrac12(1+\cos\vartheta) \qquad \text{and} \qquad \sin^2\tfrac12\vartheta = \tfrac12(1-\cos\vartheta).$$

Only the magnitude of $\cos\frac12\vartheta$ and $\sin\frac12\vartheta$ are determined by these identities, not whether they are positive or negative. The tangent of $\frac12\vartheta$ is completely determined by $\cos\vartheta$ and $\sin\vartheta$. This can be seen by manipulating the foregoing identities, or by reflecting on the figure below, and then appealing to analytic continuation.



$$\tan\tfrac12\vartheta = \frac{\sin\vartheta}{1+\cos\vartheta}$$

The use of these identities for evaluating trigonometric functions is illustrated next.

4.1. *Elementary calculations.* Suppose that $P(3,-4)$ is on the terminal side of α and $Q(-5,12)$ is on the terminal side of β (with both α and β in standard position). The distance from P to the origin is 5, and the distance from Q to the origin is 13 (by Pythagoras' formula). The double and half angle identities for the tangent give

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(-\frac43\right)}{1-\left(-\frac43\right)^2} = \tfrac{24}{7}, \quad \text{and} \quad \tan\tfrac12\beta = \frac{\sin\beta}{1+\cos\beta} = \frac{\frac{12}{13}}{1+\left(-\frac{5}{13}\right)} = \tfrac32.$$

The sum identity for the cosine function gives,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(\tfrac35\right)\left(-\tfrac{5}{13}\right) - \left(-\tfrac45\right)\left(\tfrac{12}{13}\right) = \tfrac{33}{65},$$

and the half angle identity for the sine function gives

$$\sin^2\tfrac12\alpha = \tfrac12(1-\cos\alpha) = \tfrac12\left(1-\tfrac35\right) = \tfrac15, \quad \text{so} \quad \sin\tfrac12\alpha = \pm\tfrac15\sqrt5.$$

Further determination requires more information. For example, if $\frac32\pi<\alpha<2\pi$ then $\frac34\pi<\frac12\alpha<\pi$, where the sine function is positive, so $\sin\frac12\alpha=\frac15\sqrt5$. On the other hand, if $-\frac12\pi<\alpha<0$ then $-\frac14\pi<\frac12\alpha<0$, where the sine function is negative, so $\sin\frac12\alpha=-\frac15\sqrt5$..

4.2. *Multiples of* $\frac{1}{12}\pi$. Any integer multiple of $\frac{1}{12}$ can be written the sum of an integer multiple of $\frac13$ and an integer multiple of $\frac14$. For example, $\frac{5}{12}=\frac23-\frac14$, so the difference identity for the sine function gives

$$\begin{aligned}\sin\tfrac{5}{12}\pi = \sin\left(\tfrac23\pi-\tfrac14\pi\right) &= \sin\tfrac23\pi\cos\tfrac14\pi - \cos\tfrac23\pi\sin\tfrac14\pi = \left(\tfrac12\sqrt3\right)\left(\tfrac12\sqrt2\right) - \left(-\tfrac12\right)\left(\tfrac12\sqrt2\right) \\ &= \tfrac14\sqrt2+\tfrac14\sqrt6, \quad \text{or} \quad \tfrac14(1+\sqrt3)\sqrt2.\end{aligned}$$

4.3. *Halving angles.* The half angle identities can be used to evaluate trigonometric functions of $\frac{p}{q}\pi$, where q is a power of 2, or the product of 3 and a power of 2. For example, $\cos^2\frac58\pi=\frac12(1+\cos\frac54\pi)=\frac12(1-\frac12\sqrt2)=\frac14(2-\sqrt2)$, and $\frac12\pi<\frac58\pi<\pi$ where the cosine is negative, so $\cos\frac58\pi=\frac12\sqrt{2-\sqrt2}$. To take another example, writing $\frac{1}{12}\pi$ as $\frac13\pi-\frac14\pi$, and using difference identities, gives $\cos\frac{1}{12}\pi=\frac14(1+\sqrt3)\sqrt2$ and $\sin\frac{1}{12}\pi=\frac14(\sqrt3-1)\sqrt2$. (*Exercise*: Make the calculations required to verify these results.) So the half angle identity for the tangent gives, after rationalizing the denominator and simplifying the result,

$$\tan\tfrac{1}{24}\pi = \frac{\sin\frac{1}{24}\pi}{1+\cos\frac{1}{24}\pi} = \frac{\frac14(\sqrt3-1)\sqrt2}{1+\frac14(1+\sqrt3)\sqrt2} = -2+\sqrt2-\sqrt3+\sqrt6.$$

5. **Converting products to sums.** A products of sines and/or cosines can be expressed as a sum (or, as a difference) by adding or subtracting difference and sum identities, and then solving for the product that remains on the right hand side of the result. *Exercise*: Use this technique to derive the following identities.

$$\begin{aligned}\cos\alpha\cos\beta &= \tfrac12\left\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\right\}. \\ \sin\alpha\sin\beta &= \tfrac12\left\{\cos(\alpha-\beta)-\cos(\alpha-\beta)\right\}. \\ \sin\alpha\cos\beta &= \tfrac12\left\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\right\}\end{aligned}$$

6. **Converting sums to products.** A sum (or differences) of sines, or cosines, can be expressed as a product by doubling one of the preceding identities, and then expressing the result in terms of the arguments of the summands. *Exercise*: Use this technique to derive the following identities.

$$\begin{aligned}\sin\alpha\pm\sin\beta &= 2\sin\tfrac12(\alpha\pm\beta)\cos\tfrac12(\alpha\mp\beta) \\ \cos\alpha+\cos\beta &= 2\cos\tfrac12(\alpha+\beta)\cos\tfrac12(\alpha-\beta) \\ \cos\alpha-\cos\beta &= -2\sin\tfrac12(\alpha+\beta)\sin\tfrac12(\alpha-\beta)\end{aligned}$$

7. **Rational parametrization of a circle.** The square of the half angle identity for the tangent can used to express $\cos\vartheta$ and $\sin\vartheta$ as rational functions of $z=\tan\frac12\vartheta$. *Exercise*: Show that

$$\cos\vartheta = \frac{1-z^2}{1+z^2} \qquad \text{and} \qquad \sin\vartheta = \frac{2z}{1+z^2} \qquad \text{where} \qquad z=\tan\tfrac12\vartheta.$$

This expresses points on the unit circle as rational functions of a variable z, and is of importance in the integration of rational trigonometric functions. It also implies that the integer side lengths of a right angled triangle are precisely those of the form $2pqr$, $r(p^2-q^2)$, $r(p^2+q^2)$, where p, q and r are integers.