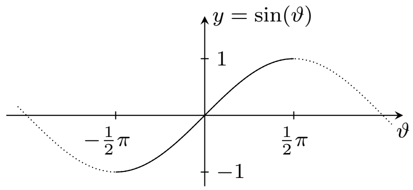
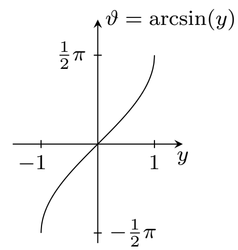
Inverse functions. If a function f maps a subset D of its domain one-to-one onto its range, then the restriction of f to D has an inverse, f^{-1}, whose domain is the range of f and whose range is D. By definition,

$$x = f^{-1}(y) \qquad \text{means that} \qquad f(x) = y \qquad \text{and} \qquad x \in D. \tag{†}$$

Thus $f(f^{-1}(y)) = y$ for all y in the range of f, but $f^{-1}(f(x)) = x$ if, and only if, $x \in D$. If D is a proper subset of the domain of f, then f^{-1} is merely a *right inverse*, or *section*, of the unrestricted function f.

The inverse sine function. The restriction of the sine function to $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ is one-to-one; its inverse is called the *inverse sine*, or *arcsine*, function, and is denoted by arcsin. The domain of arcsin is $[-1, 1]$ and the range of arcsin is $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Below are graphs of $y = \sin(\vartheta)$ and $\vartheta = \arcsin(y)$.





By definition,

$$\vartheta = \arcsin(y) \qquad \text{means that} \qquad \sin(\vartheta) = y \qquad \text{and} \qquad -\tfrac{1}{2}\pi \leqslant \vartheta \leqslant \tfrac{1}{2}\pi.$$

Differentiating the equation $\sin(\vartheta) = y$ with respect to ϑ gives

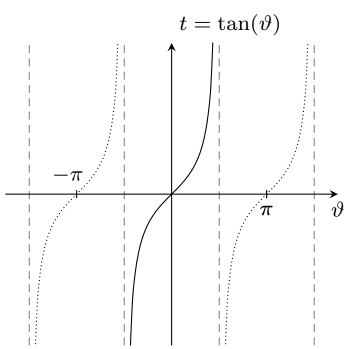
$$\frac{dy}{d\vartheta} = \cos(\vartheta), \qquad \text{or} \qquad \frac{d\vartheta}{dy} = \frac{1}{\cos(\vartheta)}, \qquad \text{provided} \qquad -\tfrac{1}{2}\pi < \vartheta < \tfrac{1}{2}\pi.$$

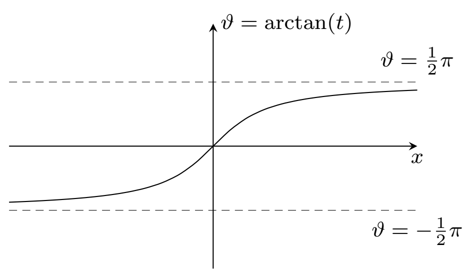
The latter restriction implies that $\cos(\vartheta) > 0$, and hence $\cos\vartheta = \sqrt{1-\sin^2(\vartheta)} = \sqrt{1-y^2}$. Therefore,

$$\frac{d}{dy}\big(\arcsin(y)\big) = \frac{1}{\sqrt{1-y^2}}, \qquad \text{for} \qquad -1 < y < 1.$$

Other symbols for the inverse sine function include asin and $\sin^{-1}$.

The inverse tangent function. The restriction of the tangent function to $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ is one-to-one, and its inverse is called the *inverse tangent*, or *arctangent*, function, which is denoted by arctan. The domain of arctan is $\mathbb{R}$ and the range of arctan is $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Below are graphs of $t = \tan(\vartheta)$ and $\vartheta = \arctan(t)$.





By definition,

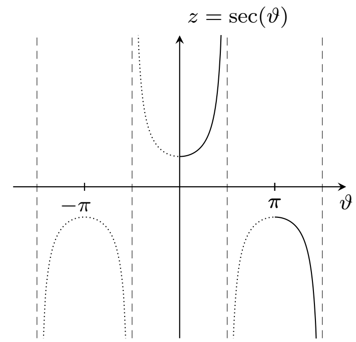
$$\vartheta = \arctan(t) \qquad \text{means that} \qquad \tan(\vartheta) = t \qquad \text{and} \qquad -\tfrac{1}{2}\pi < \vartheta < \tfrac{1}{2}\pi.$$

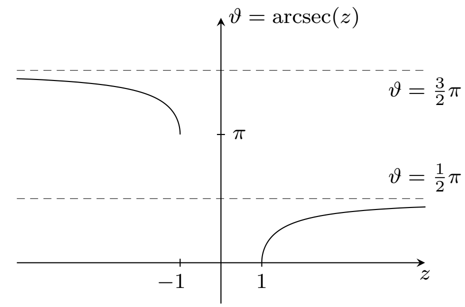
Differentiating $\tan(\vartheta) = t$ with respect to ϑ yields $\dfrac{dt}{d\vartheta} = 1 + \tan^2(\vartheta) = 1 + t^2$, or $\dfrac{d\vartheta}{dt} = \dfrac{1}{1+t^2}$. Therefore,

$$\frac{d}{dt}\big(\arctan(t)\big) = \frac{1}{1+t^2}, \qquad \text{for} \qquad t \in \mathbb{R}.$$

Other symbols for the inverse tangent function include atan and $\tan^{-1}$.

The inverse secant function. The restriction of the secant function to $[0, \frac{1}{2}\pi) \cup [\pi, \frac{3}{2}\pi)$ is one-to-one; its inverse is called the *inverse secant*, or *arcsecant*, function, and is denoted by arcsec. The domain of arcsec is $(-\infty, -1] \cup [1, \infty)$ and the range of arcsec is $[0, \frac{1}{2}\pi) \cup [\pi, \frac{3}{2}\pi)$. Below are graphs of $z = \sec(\vartheta)$, and $\vartheta = \operatorname{arcsec}(z)$.





By definition,

$$\vartheta = \operatorname{arcsec}(z) \qquad \text{means that} \qquad \sec(\vartheta) = z \qquad \text{and} \qquad 0 \leqslant \vartheta < \tfrac{1}{2}\pi \qquad \text{or} \qquad \pi \leqslant \vartheta < \tfrac{3}{2}\pi.$$

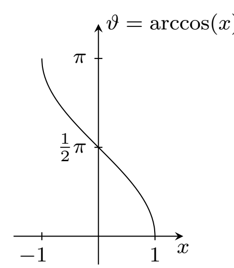
The equation $\sec(\vartheta) = z$, *together with the restriction* $0 \leqslant \vartheta < \frac{1}{2}\pi$ or $\pi \leqslant \vartheta < \frac{3}{2}\pi$, implies that

$$\frac{d}{dz}\big(\operatorname{arcsec}(z)\big) = \frac{1}{z\sqrt{z^2-1}}, \qquad \text{provided} \qquad z < -1 \quad \text{or} \quad z > 1.$$

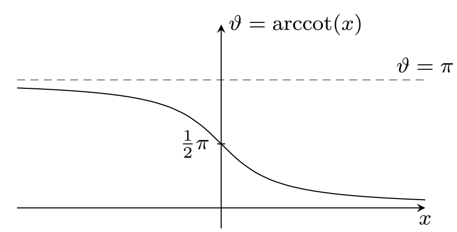
Other symbols for the inverse secant function include asec and $\sec^{-1}$.

The other inverse trigonometric functions. The remaining inverse trigonometric functions are defined using the cofunction identities for the corresponding trigonometric functions. Below are their definitions and graphs. Their domains, ranges and derivatives can be seen from their definitions (and/or their graphs).

$$\arccos(x) = \tfrac{1}{2}\pi - \arcsin(x)$$



$$\operatorname{arccot}(x) = \tfrac{1}{2}\pi - \arctan(x)$$



$$\operatorname{arccsc}(x) = \tfrac{1}{2}\pi - \operatorname{arcsec}(x)$$